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Stardom Scientific Journal of Natural and Engineering Sciences

- Stardom Scientific Journal of Natural and Engineering Sciences -
Peer Reviewed Scientific Journal published twice
a year by Stardom University

1st issue- 3rd Volume 2025

ISSN 2980-3756

The Relationship Between Weyl's Curvature Tensor and The Coharmonic Tesdor in Generalized Recurrent Finsler Spaces

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Abstract: This paper explores the geometric relationship between Weyl's curvature tensor and the conharmonic tensor in generalized recurrent Finsler spaces. The study begins with a review of the fundamental definitions and recurrence conditions governing Finsler geometry. Using tensorial identities and recurrence properties, several geometric relations are derived to describe how these tensors behave under specific recurrence transformations. The main results show that the Weyl and conharmonic tensors are interrelated through certain curvature conditions, leading to equivalence in their vanishing and recurrence properties under well-defined constraints. These findings contribute to a deeper understanding of the intrinsic geometry of generalized recurrent Finsler spaces and offer potential applications in the study of geometric structures with special curvature properties. Furthermore, the results may have implications for broader areas in differential geometry and mathematical physics, where such tensors play a key role in describing the curvature and topology of manifolds.

Keywords: Weyl's curvature tensor, conharmonic tensor, generalized recurrent Finsler spaces, differential geometry.

I. Introduction

Finsler geometry has gained significant attention due to its broad applications in differential geometry and mathematical physics. Among the various curvature tensors that characterize the geometric structure of Finsler spaces, Weyl's curvature tensor and the conharmonic tensor play essential roles in describing the intrinsic and extrinsic properties of such manifolds.

Generalized recurrent Finsler spaces, which extend the classical concept of recurrent spaces, provide a rich framework for studying the interdependence between different curvature tensors under specific recurrence conditions. Previous studies have primarily addressed individual properties of these tensors; however, the relationship between Weyl's curvature tensor and the conharmonic tensor in this generalized setting remains relatively unexplored.

This study aims to investigate the geometric relationship between these two tensors in generalized recurrent Finsler spaces. By utilizing recurrence conditions, tensorial identities, and curvature properties, new results are derived that establish equivalence conditions and interdependence between the tensors. These findings not only contribute to a deeper understanding of Finsler geometry but also provide a foundation for further studies in mathematical physics, where curvature tensors are essential in describing gravitational and geometric phenomena.

In this paper, we investigate the properties of the conharmonic curvature tensor, L^i_{jkh}, in the context of Finsler geometry. The study focuses on the generalized recurrent Finsler spaces and provides new insights into the curvature properties of these spaces, particularly those with a second-order covariant derivative. The work builds upon earlier studies, notably by Al-Qashbari, Abdallah, and Al-ssallal, and extends the concept of generalized recurrent Finsler spaces by incorporating new conditions under which the curvature tensor remains invariant under certain transformations.

Finsler geometry, an extension of Riemannian geometry, has been extensively studied for its wide applicability in both mathematics and physics, particularly in the context of spacetime curvature. A substantial body of work has focused on various properties of curvature tensors and their implications for higher-dimensional spaces. Among the key aspects of Finsler geometry, the study of recurrent structures and curvature tensors plays a pivotal role in understanding the intrinsic geometry of these spaces. Ahsan and Ali (2014) first investigated some properties of the www-curvature tensor, which serves as a foundational element in the exploration of Finsler spaces with specific curvature characteristics. In their 2016 study, Ahsan and Ali expanded on these properties, providing a deeper analysis of the curvature tensor in the context of general relativity, particularly the spacetime curvature. This led to a better understanding of the geometric structures governing the spacetime continuum and set the stage for further exploration of curvature properties in more general settings, including Finsler spaces.

In recent years, there has been a surge of interest in the study of higher-order derivatives and special curvature tensors in Finsler spaces. Abu-Donia et al. (2020) focused on the w^*-curvature tensor in relativistic space-times, exploring its role in the analysis of spacetime geometry and its physical implications. The study of special curvature tensors such as these has contributed to a better understanding of the underlying geometrical structures, particularly in the context of relativistic physics. Building on these foundational studies, Al-Qashbari et al. (2024) explored recurrent Finsler structures, providing higher-order generalizations defined by special curvature tensors. Their work introduced novel perspectives on the behavior of curvature tensors under specific constraints and contributed significantly to the development of a generalized framework for recurrent Finsler spaces. This work aligns with the broader trend of investigating higher-order derivatives, a direction pursued by Al-Qashbari and his collaborators, who have extensively studied Berwald's and Cartan's higher-order derivatives in Finsler space, demonstrating their influence on curvature tensor properties.

The ongoing exploration of decomposition analysis, such as Al-Qashbari et al.'s (2024) study of Weyl's curvature tensor via Berwald's derivatives, and the study of generalized curvature relations, continues to push the boundaries of Finsler geometry. These studies, along with the works of Misra et al. (2014), Goswami (2017), and others, have advanced our understanding of Finsler spaces and their curvature relations, especially in the context of recurrent and generalized structures.

The work of Al-Qashbari and his colleagues, including their studies on generalized recurrent Finsler spaces and various decomposition techniques, contributes to the ongoing development of Finsler geometry. Their research on the conharmonic curvature tensor and its properties in generalized Finsler spaces provides valuable insights into the intricate relationships between curvature, torsion, and the underlying geometric structures.

This paper builds on the foundation laid by previous studies, particularly focusing on the role of conharmonic curvature tensors in generalized recurrent Finsler spaces. By extending existing methods and exploring new techniques, we aim to deepen the understanding of the geometry of these spaces, offering new avenues for further research in the field.

In this paper, we investigate some identities between Weyl's tensor W^i_{jkh} and conharmonic tensor L^i_{jkh}. We first introduce the basic concepts of Weyl's curvature tensor and conharmonic tensor L^i_{jkh}. Then, we derive some identities between these two tensors.

2. Preliminaries

In this section, we provide the necessary conditions and definitions relevant to the purpose of this paper. Additionally, the two vectors y_i and y^i satisfy the following conditions:

a) $y_i = g_{ij}\, y^j$, b) $y_i\, y^i = F^2$, c) $\delta_j^k y^j = y^k$

d) $g_{ir}\,\delta_j^i = g_{rj}$ and e) $g^{jk}\delta_k^i = g^{ji}$. (2.1)

The quantities g_{ij} and g^{ij} are related as follows:

$$\text{a)}\quad g_{ij}\, g^{jk} = \delta_i^k = \begin{cases} 1 , & if \quad i = k , \\ 0 , & if \quad i \neq k . \end{cases} \tag{2.2}$$

b) $g_{ij|h} = 0$ and c) $g^{ij}{}_{|h} = 0$.

The h-covariant derivative of second order for an arbitrary vector field with respect to x^k and x^j, successively, we get

$$X^i_{|k|j} = \partial_j(X^i_{|k}) - (X^i_{|r})\Gamma^{*r}_{kj} + (X^r_{|k})\Gamma^{*i}_{rj} - (\partial_j X^i_{|k})\Gamma^{*i}_{js} y^s. \tag{2.3}$$

The vector y^i and metric function F vanish identically under Cartan's covariant derivative

a) $F_{|h} = 0$ and b) $y^i{}_{|h} = 0$. (2.4)

The tensor W^i_{jkh} , the torsion tensor W^i_{jk} and the deviation tensor W^i_j are defined as follows:

$$W^i_{jkh} = H^i_{jkh} + \frac{2\,\delta^i_j}{(n+1)} H_{[hk]} + \frac{2\,y^i}{(n+1)}\dot{\partial}_j H_{[kh]} + \frac{\delta^i_k}{(n^2-1)}\big(n\, H_{jh} + H_{hj} + y^r \dot{\partial}_j H_{hr}$$

$$- \frac{\delta^i_h}{(n^2-1)}\big(n\, H_{jk} + H_{kj} + y^r \dot{\partial}_j H_{kr}\big) , \tag{2.5}$$

$$W^i_{jk} = H^i_{jk} + \frac{y^i}{(n+1)} H_{[jk]} + 2\left\{\frac{\delta^i_{[\,j}}{(n^2-1)}\big(n\, H_{k]} - y^r H_{k]\,r}\big)\right\} , \tag{2.6}$$

and $W^i_j = H^i_j - H\delta^i_j - \frac{1}{(n+1)}\big(\dot{\partial}_r H^r_j - \dot{\partial}_j H\big)\, y^i$, respectively. (2.7)

Additionally, assuming that the tensor W^i_j satisfies the following identities

a) $W^i_k\, y^k = 0$, b) $W^i_i = 0$, c) $W^i_k\, y_i = 0$,

d) $g_{ir}\, W^i_j = W_{rj}$, e) $g^{jk} W_{jk} = W$ and f) $W_{jk}\, y^k = 0$. (2.8)

we have the conharmonic curvature tensor L^i_{jkh} , torsion tensor L^i_{jk} , Ricci tensor L_{jk} , curvature vector L_k , and scalar curvature L satisfying:

a) $L^i_{jkh}\, y^j = L^i_{kh}$, b) $L^i_{kh}\, y^k = L^i_h$, c) $L^i_{jki} = L_{jk}$

d) $L^i_{ki} = L_k$, e) $L^i_i = L$ and f) $g_{ir}\, L^i_{jkh} = L_{rjkh}$. (2.9)

The Cartan third curvature tensor R^i_{jkh} , Ricci tensor R_{jk} , the vector H_k , and the scalar curvature H are defined as:

a) $R_{jk}\, y^j = H_k$, b) $R_{jk}\, y^k = R_j$, c) $R^i_i = R$ and d) $H_k\, y^k = (n-1)H$. (2.10)

Al-Qashbari and Al-Ssallal [5], as well as Al-Qashbari, Haouse, and Al-Ssallal [6], introduced and studied the curvature tensor using Berwald's and Cartan's first and second-order derivatives in Finsler space, which are characterized by the condition:

$$W^i_{jkh|m} = \lambda_m W^i_{jkh} + \mu_m(\delta^i_k g_{jh} - \delta^i_h g_{jk}) + \tfrac{1}{4}\gamma_m(W^i_k g_{jh} - W^i_h g_{jk}) \ . \tag{2.11}$$

A Finsler space F_n , in which the curvature tensor W^i_{jkh} satisfies the condition (2.11), is referred to as the generalized $W_{|h}$-recurrent space and denoted by $G^{2nd}\, W_{|h}$- RF_n .

Taking the covariant derivative of (2.11) with respect to x^l in the context of Cartan's connection, we obtain:

$$W^i_{jkh|m|l} = (\lambda_{m|l})W^i_{jkh} + \lambda_m(W^i_{jkh|l}) + (\mu_{m|l})(\delta^i_k g_{jh} - \delta^i_h g_{jk})$$
$$+\mu_m(\delta^i_k g_{jh} - \delta^i_h g_{jk})_{|l} + \tfrac{1}{4}\gamma_m(W^i_k g_{jh} - W^i_h g_{jk})_{|l} + \tfrac{1}{4}\gamma_{m|l}(W^i_k g_{jh} - W^i_h g_{jk}). \tag{2.12}$$

By applying equations (2.2b) and (2.11) to equation (2.12), we get

$$W^i_{jkh|m|l} = \lambda_{m|l}W^i_{jkh} + \lambda_m\left(\lambda_l W^i_{jkh} + \mu_l(\delta^i_k g_{jh} - \delta^i_h g_{jk}) + \tfrac{1}{4}\gamma_l(W^i_k g_{jh} - W^i_h g_{jk})\right)$$
$$+ \mu_{m|l}(\delta^i_k g_{jh} - \delta^i_h g_{jk}) + \tfrac{1}{4}\gamma_{m|l}(W^i_k g_{jh} - W^i_h g_{jk}) + \tfrac{1}{4}\gamma_m(W^i_k g_{jh} - W^i_h g_{jk})_{|l} \ .$$

Or

$$W^i_{jkh|m|l} = (\lambda_{m|l} + \lambda_m\lambda_l)W^i_{jkh} + (\mu_{m|l} + \lambda_m\mu_l)(\delta^i_k g_{jh} - \delta^i_h g_{jk})$$
$$+\tfrac{1}{4}\gamma_m(W^i_k g_{jh} - W^i_h g_{jk})_{|l} + \tfrac{1}{4}(\lambda_m\gamma_l + \gamma_{m|l})(W^i_k g_{jh} - W^i_h g_{jk}). \tag{2.13}$$

The equation (2.13), can be expressed as:

$$W^i_{jkh|m|l} = a_{ml}W^i_{jkh} + b_{ml}(\delta^i_k g_{jh} - \delta^i_h g_{jk}) + \tfrac{1}{4}c_{ml}(W^i_k g_{jh} - W^i_h g_{jk})$$
$$+\tfrac{1}{4}\gamma_m(W^i_k g_{jh} - W^i_h g_{jk})_{|l} \, . \tag{2.14}$$

where $a_{ml} = \lambda_{m|l} + \lambda_m\lambda_l$, $b_{ml} = \mu_{m|l} + \lambda_m\mu_l$ and $c_{ml} = (\lambda_m\gamma_l + \gamma_{m|l})$ are non-zero covariant tensors field of second order and γ_m is non-zero covariant victor of first order, respectively.

Definition 2.1. In Finsler space, which the Wely's projective curvature tensor W^i_{jkh} satisfies the condition (2.14) is called the generalization generalized $W_{|h}$-birecurrent space and the tensor will be called a generalization generalized h-birecurrent space. These space and tensor denote them briefly by $G^{2nd}W_{|h}$-BRF_n and G^{2nd}h- BR , respectively.

We consider an n-dimensional Finsler space F_n , the Weyls projective curvature tensor W^i_{jkh} satisfies the condition (2.11) and (2.14), These spaces denoted by $G^{2nd}W_{|h}$-RF_n and
$G^{2nd}W_{|h}$-BRF_n , respectively.

3. Relationship Between Wely's Curvature Tensor W^i_{jkh} and Conharmonic Tensor L^i_{jkh}

Finsler geometry, as a generalization of Riemannian geometry, provides a powerful framework for modeling a wide range of physical phenomena. In Finsler spaces, the curvature properties of the space are characterized by various curvature tensors, among which Weyl and the conharmonic tensor L^i_{jkh} play a significant role. While the geometric interpretations and physical implications of these tensors have been extensively studied, the relationship between them remains a subject of ongoing research. This paper aims to investigate the connection between Weyl's curvature tensor and the conharmonic tensor L^i_{jkh} in Finsler spaces. By exploring their algebraic and geometric properties, we seek to establish new identities and inequalities that relate these two tensors. Our findings are expected to contribute to a deeper understanding of the curvature structure of Finsler spaces and provide insights into their applications in physics, such as in the study of gravitational theories and cosmology.

Some properties of W^i_{jkh} curvature tensor was proposed by Al-Qashbari, Abdallah and Al-ssallal. For $(n = 4)$ a Riemannian space, Weyl defined the conharmonic tensor L^i_{jkh} often known as the Weyl conharmonic tensor, as

$$W^i_{jkh} = L^i_{jkh} + \tfrac{1}{2}(\delta^i_h R_{jk} - g_{jh} R^i_k) - \tfrac{1}{6}(\delta^i_k R_{jh} - g_{jk} R^i_h) \ . \qquad (3.1)$$

By taking the $h-$ covariant derivative of (3.1), with respect to x^m, we obtain:

$$W^i_{jkh|m} = L^i_{jkh|m} + \tfrac{1}{2}(\delta^i_h R_{jk} - g_{jh} R^i_k)_{|m} - \tfrac{1}{6}(\delta^i_k R_{jh} - g_{jk} R^i_h)_{|m} \ . \qquad (3.2)$$

Using (2.2b), in the equation (3.2) can be written as

$$W^i_{jkh|m} = L^i_{jkh|m} + \tfrac{1}{2}(\delta^i_h R_{jk|m} - g_{jh} R^i_{k|m}) - \tfrac{1}{6}(\delta^i_k R_{jh|m} - g_{jk} R^i_{h|m}) \ . \qquad (3.3)$$

By substituting equations (2.11) and (3.1) into (3.3), we obtain:

$$\begin{aligned} &L^i_{jkh|m} + \tfrac{1}{2}(\delta^i_h R_{jk|m} - g_{jh} R^i_{k|m}) - \tfrac{1}{6}(\delta^i_k R_{jh|m} - g_{jk} R^i_{h|m}) \\ &= \lambda_m L^i_{jkh} + \tfrac{1}{2}\lambda_m(\delta^i_h R_{jk} - g_{jh} R^i_k) - \tfrac{1}{6}\lambda_m(\delta^i_k R_{jh} - g_{jk} R^i_h) + \mu_m(\delta^i_k g_{jh} - \delta^i_h g_{jk}) \\ &+ \tfrac{1}{4}\gamma_m(W^i_k g_{jh} - W^i_h g_{jk}) \ . \qquad (3.4) \end{aligned}$$

The equation (3.4), can be expressed as:

$$\begin{aligned} L^i_{jkh|m} &= \lambda_m L^i_{jkh} + \mu_m(\delta^i_k g_{jh} - \delta^i_h g_{jk}) + \tfrac{1}{4}\gamma_m(W^i_k g_{jh} - W^i_h g_{jk}) \\ &- \tfrac{1}{2}(\delta^i_h R_{jk} - g_{jh} R^i_k)_{|m} + \tfrac{1}{6}(\delta^i_k R_{jh} - g_{jk} R^i_h)_{|m} + \tfrac{1}{2}\lambda_m(\delta^i_h R_{jk} - g_{jh} R^i_k) \\ &- \tfrac{1}{6}\lambda_m(\delta^i_k R_{jh} - g_{jk} R^i_h) \ . \qquad (3.5) \end{aligned}$$

Definition of the Space

The above equation is defined on an n-dimensional differentiable Riemannian manifold (M, g) equipped with a Levi-Civita connection and the associated curvature tensors.

All tensorial quantities appearing in the equation such as g_{ij} , δ^i_j, R_{ij}, $R^i{}_j$, and $L^i{}_{jkh}$ are smooth tensor fields on the manifold.

The symbol $(\cdot)_{|m}$ denotes covariant differentiation with respect to the Levi-Civita connection.
The functions λ_m, μ_m, and γ_m are smooth covector fields on M.
Thus, the equation is formulated entirely in the tensor algebra of the Riemannian manifold (M, g).
This demonstrates that

$$L^i_{jkh|m} = \lambda_m L^i_{jkh} + \mu_m(\delta^i_k g_{jh} - \delta^i_h g_{jk}) + \frac{1}{4}\gamma_m(W^i_k g_{jh} - W^i_h g_{jk}) . \tag{3.6}$$

If and only if

$$(\delta^i_h R_{jk} - g_{jh} R^i_k)_{|m} = \lambda_m(\delta^i_h R_{jk} - g_{jh} R^i_k) ,$$

and $(\delta^i_k R_{jh} - g_{jk} R^i_h)_{|m} = \lambda_m(\delta^i_k R_{jh} - g_{jk} R^i_h)$. (3.7)

In conclusion the proof of theorem is completed, we can determine

Theorem 3.1. In the space $G^{2nd}L_{|h}$-RF_n , the conharmonic curvature tensor L^i_{jkh} represents a generalized recurrent Finsler space, provided that the condition (3.7) is satisfied.
By transvecting equation (3.5) with y^j and utilizing equations (2.9a), (2.4b), (2.1a) and (2.10a), we obtain the following result

$$L^i_{kh|m} = \lambda_m L^i_{kh} + \mu_m(\delta^i_k y_h - \delta^i_h y_k) + \frac{1}{4}\gamma_m(W^i_k y_h - W^i_h y_k) - \frac{1}{2}(\delta^i_h H_k - y_h R^i_k)_{|m}$$
$$+\frac{1}{6}(\delta^i_k H_h - y_k R^i_h)_{|m} + \frac{1}{2}\lambda_m(\delta^i_h H_k - y_h R^i_k) - \frac{1}{6}\lambda_m(\delta^i_k H_h - y_k R^i_h) . \tag{3.8}$$

This demonstrates that

$$L^i_{kh|m} = \lambda_m L^i_{kh} + \mu_m(\delta^i_h y_k - \delta^i_k y_h) + \frac{1}{4}\gamma_m(W^i_k y_h - W^i_h y_k) . \tag{3.9}$$

If and only if

$$(\delta^i_h H_k - y_h R^i_k)_{|m} = \lambda_m(\delta^i_h H_k - y_h R^i_k) ,$$

and $(\delta^i_k H_h - y_k R^i_h)_{|m} = \lambda_m(\delta^i_k H_h - y_k R^i_h)$. (3.10)

Therefore, the proof of theorem is completed, we can say
Theorem 3.2. In the space $G^{2nd}L_{|h}$-RF_n , the torsion tensor L^i_{kh} (Conharmonic curvature tensor L^i_{jkh}) represents a generalized recurrent Finsler space, provided that the condition (3.10) is satisfied.
By transvecting equation (3.8) with y^k and utilizing equations $n = 4$, (2.9b), (2.4b), (2.1b), (2.8a), (2.1c) and (2.10d), we obtain the following result

$$L^i_{h|m} = \lambda_m L^i_h + \mu_m(y^i y_h - \delta^i_h F^2) - \frac{1}{4}\gamma_m(W^i_h F^2) - \frac{1}{2}(3\delta^i_h H - y_h R^i_k y^k)_{|m}$$
$$+\frac{1}{6}(y^i H_h - F^2 R^i_h)_{|m} + \frac{1}{2}\lambda_m(3\delta^i_h H - y_h R^i_k y^k) - \frac{1}{6}\lambda_m(y^i H_h - F^2 R^i_h). \tag{3.11}$$

This demonstrates that

$$L^i_{h|m} = \lambda_m L^i_h + \mu_m(\delta^i_h F^2 - y^i y_h) + \frac{1}{4}\gamma_m(W^i_h F^2) . \tag{3.12}$$

If and only if

$$\left(3\delta_h^i\, H - y_h R_k^i\, y^k\right)_{|m} = \lambda_m\left(3\delta_h^i\, H - y_h R_k^i\, y^k\right) \;,$$

and $\left(y^i H_h - F^2 R_h^i\right)_{|m} = \lambda_m\left(y^i H_h - F^2 R_h^i\right)$. (3.13)

Therefore, the proof of theorem is completed, we can say

Theorem 3.3. In the space $G^{2nd}L_{|h}\text{-}RF_n$, deviation tensor L_h^i represents a generalized recurrent Finsler space if the tensors $\left(3\delta_h^i\, H - y_h R_k^i\, y^k\right)$ and $\left(y^i H_h - F^2 R_h^i\right)$ are generalized recurrent Finsler spaces.

By contracting the indices i and h in equations (3.5), (3.8), and (3.11), and utilizing equations (n=4), (2.2a), (2.1a), (2.1b), (2.8b), (2.8c), (2.8d), (2.10c), (2.10d) and (2.1c), in conjunction with (2.9c), (2.9d), and (2.9e), we obtain the following result:

$$L_{jk|m} = \lambda_m L_{jk} + (1-n)\mu_m\, g_{jk} + \tfrac{1}{4}\gamma_m\left(W_{jk}\right) - \tfrac{1}{2}(n-1)R_{jk|m} + \tfrac{1}{6}\left(R_{jk} - g_{jk}\, R\right)_{|m}$$

$$+\tfrac{1}{2}(n-1)\lambda_m R_{jk} - \tfrac{1}{6}\lambda_m\left(R_{jk} - g_{jk}\, R\right). \quad (3.14)$$

This demonstrates that

$$L_{jk|m} = \lambda_m\, L_{jk} + (1-n)\mu_m g_{jk} + \tfrac{1}{4}\gamma_m\left(W_{jk}\right). \quad (3.15)$$

If and only if

$$R_{jk|m} = \lambda_m R_{jk}$$

and $\left(R_{jk} - g_{jk}\, R\right)_{|m} = \lambda_m\left(R_{jk} - g_{jk}\, R\right)$. (3.16)

and $L_{k|m} = \lambda_m L_k + (1-n)\mu_m\, y_k - \tfrac{1}{2}\left(nH_k - y_i R_k^i\right)_{|m} + \tfrac{1}{6}\left(H_k - y_k R\right)_{|m}$

$$+\tfrac{1}{2}\lambda_m\left(nH_k - y_i R_k^i\right) - \tfrac{1}{6}\lambda_m\left(H_k - y_k R\right) \;. \quad (3.17)$$

This demonstrates that

$$L_{k|m} = \lambda_m L_k + (1-n)\mu_m\, y_k \;. \quad (3.18)$$

If and only if

$$\left(nH_k - y_i R_k^i\right)_{|m} = \tfrac{1}{2}\lambda_m\left(nH_k - y_i R_k^i\right)$$

and $\left(H_k - y_k R\right)_{|m} = \lambda_m\left(H_k - y_k R\right)$. (3.19)

In the last

$$L_{|m} = \lambda_m L + (1-n)\mu_m F^2 - \tfrac{1}{2}\left(3nH - y_i R_k^i\, y^k\right)_{|m} + \tfrac{1}{6}\left(3H - F^2 R\right)_{|m}$$

$$+\tfrac{1}{2}\lambda_m\left(3nH - y_h R_k^i\, y^k\right) - \tfrac{1}{6}\lambda_m\left(y^i H_i - F^2 R\right). \quad (3.20)$$

This demonstrates that

$$L_{|m} = \lambda_m L + (1-n)\mu_m F^2. \quad (3.21)$$

If and only if

$$\left(3nH - y_i R^i_k\, y^k\right)_{|m} = \lambda_m\left(3nH - y_h R^i_k\, y^k\right) ,$$

and $(3H - F^2R)_{|m} = \lambda_m(3\mathrm{H} - F^2R)$. (3.22)

In conclusion the proof of theorem is completed, we can say

Theorem 3.4. In the space $G^{2nd}L_{|h}$-RF_n , the Ricci tensor L_{jk} , vector L_k and scalar L are defined in equations (3.15), (3.18), and (3.21), respectively, if and only if the conditions in equations (3.16), (3.19), and (3.22) are satisfied.

By transvecting equations (3.5) with g_{ir} and utilizing equations (2.1d), (2.2b), (2.2c), (2.8d), and (2.9f), we obtain the following result

$$L_{rjkh|m} = \lambda_m L_{rjkh} + \mu_m\left(g_{rk}g_{jh} - g_{rh}g_{jk}\right) + \tfrac{1}{4}\gamma_m\left(W_{rk}g_{jh} - W_{rh}g_{jk}\right)$$
$$-\tfrac{1}{2}\left(g_{rh}R_{jk} - g_{jh}R_{rk}\right)_{|m} + \tfrac{1}{6}\left(g_{rk}R_{jh} - g_{jk}R_{rh}\right)_{|m} + \tfrac{1}{2}\lambda_m\left(g_{rh}R_{jk} - g_{jh}R_{rk}\right)$$
$$-\tfrac{1}{6}\lambda_m\left(g_{rk}R_{jh} - g_{jk}R_{rh}\right) . \quad (3.23)$$

This demonstrates that

$$L_{rjkh|m} = \lambda_m L_{rjkh} + \mu_m\left(g_{rk}g_{jh} - g_{rh}g_{jk}\right) + \tfrac{1}{4}\gamma_m\left(W_{rk}g_{jh} - W_{rh}g_{jk}\right) . \quad (3.24)$$

If and only if

$$\left(g_{rh}R_{jk} - g_{jh}R_{rk}\right)_{|m} = \lambda_m\left(g_{rh}R_{jk} - g_{jh}R_{rk}\right) ,$$

and $\left(g_{rk}R_{jh} - g_{jk}R_{rh}\right)_{|m} = \lambda_m\left(g_{rk}R_{jh} - g_{jk}R_{rh}\right)$. (3.25)

Therefore, the proof of theorem is completed, we can say

Theorem 3.5. In the space $G^{2nd}L_{|h}$-RF_n , the associate tensor L_{jrkh} (Conharmonic curvature tensor L^i_{jkh}) represents a generalized recurrent Finsler space if the condition in equation (3.25) is satisfied.

We introduce a new class of Finsler spaces, namely, generalized-$L_{|h}$-birecurrent spaces. These spaces extend the concept of birecurrence to a broader context and exhibit interesting geometric properties. In this study, we analyze the curvature tensor of these spaces and establish several characterization theorems. Specifically, we define $\mathcal{B}_m\mathcal{B}_l$ as the covariant derivative of second order.

By taking the h − covariant derivative of equation (3.1) with respect to x^m and x^l, respectively, we obtain the following result.

$$W^i_{jkh|m|l} = L^i_{jkh|m|l} + \tfrac{1}{2}\left(\delta^i_h R_{jk} - g_{jh}R^i_k\right)_{|m|l} + \tfrac{1}{6}\left(\delta^i_k R_{jh} - g_{jk}R^i_h\right)_{|m|l} . \quad (3.26)$$

Equation (3.26) can be rewritten as follows.

$$W^i_{jkh|m|l} = L^i_{jkh|m|l} + \tfrac{1}{2}\left(\delta^i_h R_{jk|m|l} - g_{jh}R^i_{k|m|l}\right) + \tfrac{1}{6}\left(\delta^i_k R_{jh|m|l} - g_{jk}R^i_{h|m|l}\right) . \quad (3.27)$$

Similarly, by applying equations (2.14) and (3.1) in (3.27), we obtain the result

$$L^i_{jkh|m|l} + \tfrac{1}{2}\left(\delta^i_h R_{jk|m|l} - g_{jh} R^i_{k|m|l}\right) + \tfrac{1}{6}\left(\delta^i_k R_{jh|m|l} - g_{jk} R^i_{h|m|l}\right)$$
$$= a_{ml} L^i_{jkh} + \tfrac{1}{2} a_{ml}\left(\delta^i_h R_{jk} - g_{jh} R^i_k\right) + \tfrac{1}{6} a_{ml}\left(\delta^i_k R_{jh} - g_{jk} R^i_h\right) + b_{ml}\left(\delta^i_k g_{jh} - \delta^i_h g_{jk}\right)$$
$$+ \tfrac{1}{4} c_{ml}\left(W^i_k g_{jh} - W^i_h g_{jk}\right) + \tfrac{1}{4}\gamma_m\left(W^i_k g_{jh} - W^i_h g_{jk}\right)_{|l}.$$

Alternatively, this can be expressed as:

$$L^i_{jkh|m|l} = a_{ml} L^i_{jkh} + b_{ml}\left(\delta^i_k g_{jh} - \delta^i_h g_{jk}\right) + \tfrac{1}{4} c_{ml}\left(W^i_k g_{jh} - W^i_h g_{jk}\right)$$
$$+ \tfrac{1}{4}\gamma_m\left(W^i_k g_{jh} - W^i_h g_{jk}\right)_{|l} - \tfrac{1}{2}\left(\delta^i_h R_{jk} - g_{jh} R^i_k\right)_{|m|l} - \tfrac{1}{6}\left(\delta^i_k R_{jh} - g_{jk} R^i_h\right)_{|m|l}$$
$$+ \tfrac{1}{2} a_{ml}\left(\delta^i_h R_{jk} - g_{jh} R^i_k\right) + \tfrac{1}{6} a_{ml}\left(\delta^i_k R_{jh} - g_{jk} R^i_h\right). \tag{3.28}$$

This demonstrates that

$$L^i_{jkh|m|l} = a_{ml} L^i_{jkh} + b_{ml}\left(\delta^i_k g_{jh} - \delta^i_h g_{jk}\right) + \tfrac{1}{4} c_{ml}\left(W^i_k g_{jh} - W^i_h g_{jk}\right)$$
$$+ \tfrac{1}{4}\gamma_m\left(W^i_k g_{jh} - W^i_h g_{jk}\right)_{|l}. \tag{3.29}$$

If and only if

$$\left(\delta^i_h R_{jk} - g_{jh} R^i_k\right)_{|m|l} = a_{ml}\left(\delta^i_h R_{jk} - g_{jh} R^i_k\right),$$

and $\left(\delta^i_k R_{jh} - g_{jk} R^i_h\right)_{|m|l} = a_{ml}\left(\delta^i_k R_{jh} - g_{jk} R^i_h\right).$ (3.30)

In conclusion the proof of theorem is completed, we can determine

Theorem 3.6. In the space $G^{2nd}L_{|h}$-BRF_n , the conharmonic curvature tensor L^i_{jkh} defines a generalized birecurrent Finsler space if and only if the condition in equation (3.30) is satisfied.

By transvecting condition (3.28) with y^j, and utilizing equations (2.9a), (2.4b), (2.1a), (2.1c) and (2.10a), we obtain the following result.

$$L^i_{kh|m|l} = a_{ml} L^i_{kh} + b_{ml}\left(\delta^i_k y_h - \delta^i_h y_k\right) + \tfrac{1}{4} c_{ml}\left(W^i_k y_h - W^i_h y_k\right)$$
$$- \tfrac{1}{2}\left(\delta^i_h H_k - y_h R^i_k\right)_{|m|l} - \tfrac{1}{6}\left(\delta^i_k H_h - y_k R^i_h\right)_{|m|l} + \tfrac{1}{2} a_{ml}\left(\delta^i_h H_k - y_h R^i_k\right)$$
$$+ \tfrac{1}{6} a_{ml}\left(\delta^i_k H_h - y_k R^i_h\right) + \tfrac{1}{4}\gamma_m\left(W^i_k y_h - W^i_h y_k\right)_{|l}. \tag{3.31}$$

This demonstrates that

$$L^i_{kh|m|l} = a_{ml} L^i_{kh} + b_{ml}\left(\delta^i_k y_h - \delta^i_h y_k\right) + \tfrac{1}{4} c_{ml}\left(W^i_k y_h - W^i_h y_k\right)$$
$$+ \tfrac{1}{4}\gamma_m\left(W^i_k y_h - W^i_h y_k\right)_{|l}. \tag{3.32}$$

If and only if

$$\left(\delta^i_h H_k - y_h R^i_k\right)_{|m|l} = a_{ml}\left(\delta^i_h H_k - y_h R^i_k\right),$$

and $\left(\delta^i_k H_h - y_k R^i_h\right)_{|m|l} = a_{ml}\left(\delta^i_k H_h - y_k R^i_h\right).$ (3.33)

Therefore, the proof of theorem is completed, we can say

Theorem 3.7. In the space $G^{2nd}L_{|h}$-BRF_n , the $h-$covariant derivative of second-order for the torsion tensor L^i_{kh} (Conharmonic curvature tensor L^i_{jkh}) defines a generalized birecurrent Finsler space if and only if the condition in equation (3.33) is satisfied.
By transvecting condition (3.31) with y^k, and applying $(n=4)$, along with equations (2.9b), (2.4a), (2.4b), (2.1b), (2.8a), (2.1c), and (2.10d), we obtain the following result.

$$L^i_{h|m|l} = a_{ml}L^i_h + b_{ml}\left(y^i y_h - \delta^i_h F^2\right) - \tfrac{1}{4}c_{ml}\left(W^i_h F^2\right) - \tfrac{1}{4}\gamma_m\left(W^i_h F^2\right)_{|l}$$
$$-\tfrac{1}{2}\left(3\delta^i_h H - y_h R^i_k y^k\right)_{|m|l} - \tfrac{1}{6}\left(y^i H_h - F^2 R^i_h\right)_{|m|l} + \tfrac{1}{2}a_{ml}\left(3\delta^i_h H - y_h R^i_k y^k\right)$$
$$+\tfrac{1}{6}a_{ml}\left(y^i H_h - F^2 R^i_h\right). \tag{3.34}$$

This demonstrates that

$$L^i_{h|m|l} = a_{ml}L^i_h + b_{ml}\left(y^i y_h - \delta^i_h F^2\right) - \tfrac{1}{4}c_{ml}\left(W^i_h F^2\right) - \tfrac{1}{4}\gamma_m\left(W^i_h F^2\right)_{|l}. \tag{3.35}$$

If and only if

$$\left(3\delta^i_h H - y_h R^i_k y^k\right)_{|m|l} = a_{ml}\left(3\delta^i_h H - y_h R^i_k y^k\right),$$

and $\left(y^i H_h - F^2 R^i_h\right)_{|m|l} = a_{ml}\left(y^i H_h - F^2 R^i_h\right)$. (3.36)

Therefore, the proof of theorem is completed, we can say

Theorem 3.8. In the space $G^{2nd}L_{|h}$-BRF_n , the projective deviation tensor L^i_h represents a generalized birecurrent Finsler space if the tensors $\left(3\delta^i_h H - y_h R^i_k y^k\right)$ and $\left(y^i H_h - F^2 R^i_h\right)$ are generalized birecurrent Finsler spaces.
By contracting the indices i and h in equations (3.28), (3.31), and (3.34), and utilizing equations (n=4), (2.2a), (2.1a), (2.1b), (2.8b), (2.8c), (2.8d), (2.10c), (2.10d) and (2.1c), along with the relations in equations (2.9c), (2.9d), and (2.9e), we obtain the following result.

$$L_{jk|m|l} = a_{ml}L_{jk} + (n-1)b_{ml}\,g_{jk} + \tfrac{1}{4}c_{ml}W_{jk} + \tfrac{1}{4}\gamma_m W_{jk|l} - \tfrac{1}{2}(1-n)R_{jk|m|l}$$
$$-\tfrac{1}{6}\left(R_{jk} - g_{jk}R\right)_{|m|l} + \tfrac{1}{2}(1-n)a_{ml}R_{jk} + \tfrac{1}{6}a_{ml}\left(R_{jk} - g_{jk}R\right). \tag{3.37}$$

This demonstrates that

$$L_{jk|m|l} = a_{ml}L_{jk} + (n-1)b_{ml}\,g_{jk} + \tfrac{1}{4}c_{ml}W_{jk} + \tfrac{1}{4}\gamma_m W_{jk|l}. \tag{3.38}$$

If and only if

$$R_{jk|m|l} = a_{ml}R_{jk},$$
$$\left(R_{jk} - g_{jk}R\right)_{|m|l} = a_{ml}\left(R_{jk} - g_{jk}R\right). \tag{3.39}$$

And

$$L_{k|m|l} = a_{ml}L_k + (1-n)b_{ml}\,y_k - \tfrac{1}{2}\left(nH_k - y_i R^i_k\right)_{|m|l} - \tfrac{1}{6}\left(H_k - y_k R\right)_{|m|l}$$
$$+\tfrac{1}{2}a_{ml}\left(nH_k - y_i R^i_k\right) + \tfrac{1}{6}a_{ml}\left(H_k - y_k R\right). \tag{3.40}$$

This demonstrates that

$$L_{k|m|l} = a_{ml} L_k + (1-n) b_{ml}\, y_k \;. \tag{3.41}$$

If and only if

$$\left(nH_k - y_i R^i_k\right)_{|m|l} = a_{ml}\left(nH_k - y_i R^i_k\right) \;,$$

and $\left(H_k - y_k R\right)_{|m|l} = a_{ml}\left(H_k - y_k R\right)$. (3.42)

In the last

$$L_{|m|l} = a_{ml} L + (n-1) b_{ml} F^2 - \tfrac{1}{2}\left(3nH - y_i R^i_k\, y^k\right)_{|m|l} - \tfrac{1}{6}\left(3\mathrm{H} - F^2 R\right)_{|m|l}$$
$$+ \tfrac{1}{2} a_{ml}\left(3nH - y_i R^i_k\, y^k\right) + \tfrac{1}{6} a_{ml}\left(y^i H_i - F^2 R\right) \;. \tag{3.43}$$

This demonstrates that

$$L_{|m|l} = a_{ml} L + (n-1) b_{ml} F^2 \;. \tag{3.44}$$

If and only if

$$\left(3nH - y_i R^i_k\, y^k\right)_{|m|l} = a_{ml}\left(3nH - y_i R^i_k\, y^k\right) \;,$$

and $\left(3\mathrm{H} - F^2 R\right)_{|m|l} = a_{ml}\left(3H - F^2 R\right)$. (3.45)

Therefore, the proof of theorem is completed, we can say

Theorem 3.9. In the space $\mathrm{G^{2nd}L_{|h}\text{-}BRF_n}$, the Ricci tensor L_{jk} , the vector L_k and the scalar L are defined in equations (3.38), (3.41), and (3.44), respectively, provided that the conditions (3.39), (3.42), and (3.45) are satisfied.

By transvecting equation (3.28) with $\mathrm{g_{ir}}$ and applying equations (2.1d), (2.2c), (2.8d), and (2.9f), we obtain the following result:

$$L_{rjkh|m|l} = a_{ml} L_{rjkh} + b_{ml}\left(g_{rk}\, g_{jh} - g_{rh} g_{jk}\right) + \tfrac{1}{4} c_{ml}\left(W_{rk} g_{jh} - W_{rh} g_{jk}\right)$$
$$+ \tfrac{1}{4}\, \gamma_m\left(W_{rk} g_{jh} - W_{rh} g_{jk}\right)_{|l} - \tfrac{1}{2}\left(g_{rh} R_{jk} - g_{jh} R_{rk}\right)_{|m|l} - \tfrac{1}{6}\left(g_{rk} R_{jh} - g_{jk} R_{rh}\right)_{|m|l}$$
$$+ \tfrac{1}{2} a_{ml}\left(g_{rh} R_{jk} - g_{jh} R_{rk}\right) + \tfrac{1}{6} a_{ml}\left(g_{rk} R_{jh} - g_{jk} R_{rh}\right). \tag{3.46}$$

This demonstrates that

$$L_{rjkh|m|l} = a_{ml} L_{rjkh} + b_{ml}\left(g_{rk}\, g_{jh} - g_{rh} g_{jk}\right) + \tfrac{1}{4} c_{ml}\left(W_{rk} g_{jh} - W_{rh} g_{jk}\right)$$
$$+ \tfrac{1}{4}\, \gamma_m\left(W_{rk} g_{jh} - W_{rh} g_{jk}\right)_{|l}. \tag{3.47}$$

If and only if

$$\left(g_{rh} R_{jk} - g_{jh} R_{rk}\right)_{|m|l} = a_{ml}\left(g_{rh} R_{jk} - g_{jh} R_{rk}\right) \;,$$

and $\left(g_{rk} R_{jh} - g_{jk} R_{rh}\right)_{|m|l} = a_{ml}\left(g_{rk} R_{jh} - g_{jk} R_{rh}\right)$. (3.48)

Therefore, the proof of theorem is completed, we can say

Theorem 3.10. In the space $G^{2nd}L_{|h}$-BRF_n, the associate tensor L_{jrkh} (Conharmonic curvature tensor L^i_{jkh}) characterizes a generalized birecurrent Finsler space, if condition (3.48) is satisfied.

4. Conclusions

In this study, we investigated the geometric relationship between Weyl's curvature tensor and the conharmonic tensor within the framework of generalized recurrent Finsler spaces. By applying recurrence conditions and tensorial identities, several new relations were established, leading to a better understanding of the intrinsic curvature properties of these spaces.

The main findings indicate that the behavior of Weyl's tensor and the conharmonic tensor is deeply interconnected under specific geometric constraints, resulting in conditions for their equivalence and vanishing properties. These results provide significant insights into the structure of generalized recurrent Finsler geometry and contribute to the broader study of curvature theory in differential geometry.

Future work may focus on extending these results to other curvature tensors, exploring their applications in mathematical physics, and investigating higher-order recurrence conditions to uncover further geometric properties.

5. Recommendations

Based on the results of this research, we recommend the following directions for future research:

1. Explore other types of decomposition: Investigate different decomposition schemes and their corresponding geometric interpretations.
2. Investigate the physical implications: Explore the physical implications of the decomposition results, particularly in the context of field theories and cosmology.
3. Develop numerical methods: Develop numerical methods for computing the decomposed tensors and analyzing their properties.

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تصدر بشكل نصف سنوي عن جامعة ستاردوم

العدد الأول - المجلد الثالث 2025م

رقم الإيداع الدولي: ISSN 2980-3756