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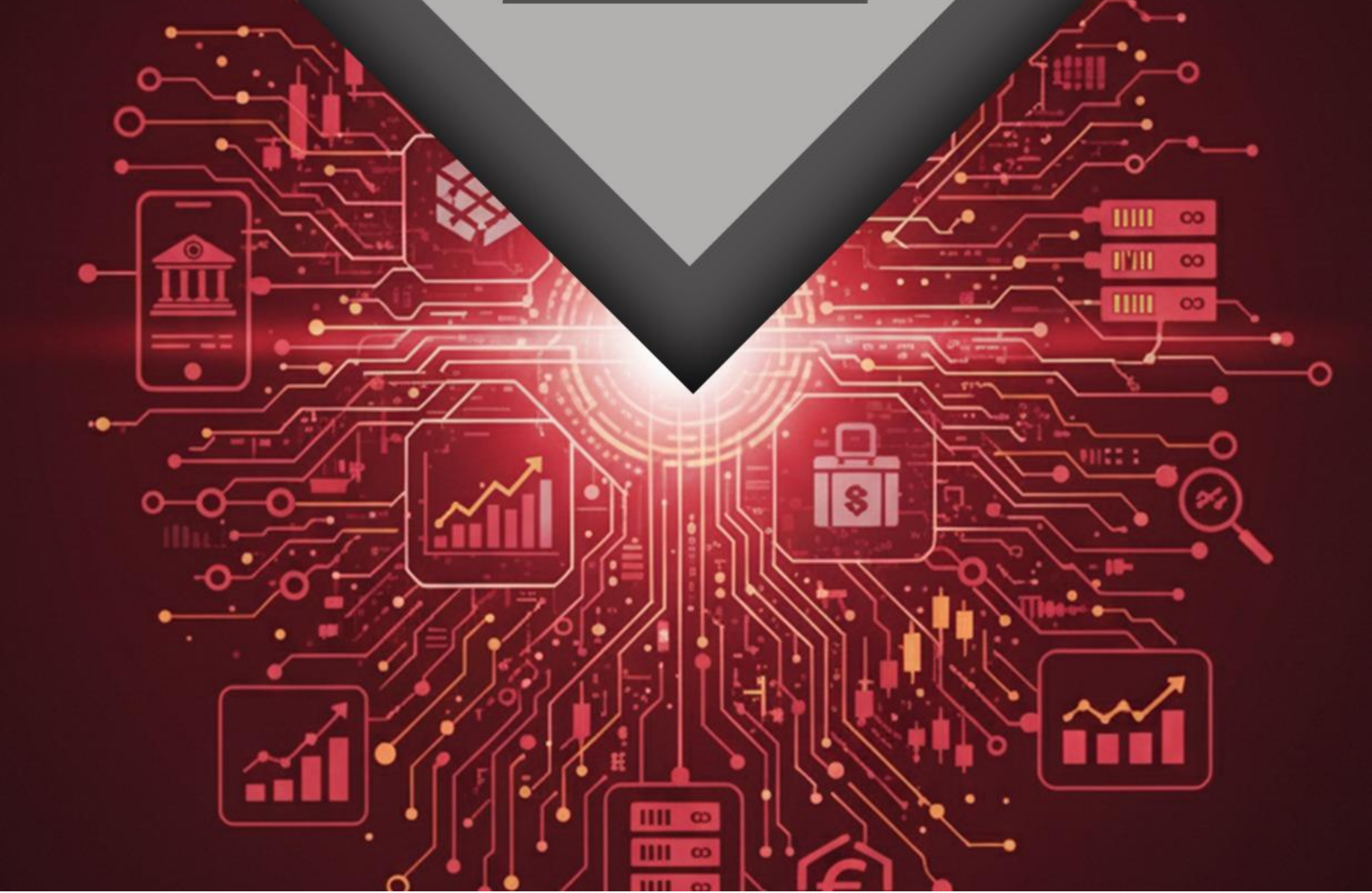

A New Extension of Beta Function by Associated with Appell Function

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Abstract.

The main objective of this paper is to introduce a further extension of extended beta function by considering Appell functions. We investigate various properties of this newly defined beta function such as integral representations, summation formulas, generalized Gauss and confluent hypergeometric functions and differentiation formulas for the extended hypergeometric and confluent hypergeometric functions. Further, some new relations for various forms of extended beta functions are obtained as special cases of the main results.

Keywords: Beta function, Appell function, Gauss hypergeometric function, confluent hypergeometric functions, Integral representations, Summation formulas, Transformation formulas.

Nomenclature:

$B(z_1, z_2)$	Classical Beta function.
$\Gamma(z)$	Gamma function.
$(\lambda)_n$	Pochhammer symbol.
${}_2F_1(a, c; d; z)$	Gauss hypergeometric function.
$\Phi(\mathrm{b}; \mathrm{c}; \mathrm{z})$	Confluent hypergeometric functions.
${}^{F_i}B_{p,q}^{(u,v)}(x, y)$	Extensions of beta function using Appell series, for $i =$ 1,2,3,4.
${}^{F_1}F_{p,q}^{(u,v)}(\delta, \gamma; \rho; z)$	New extended hypergeometric function.
${}^{F_1}\boldsymbol{\Phi}_{p,q}^{(u,v)}(\boldsymbol{\gamma}; \boldsymbol{\rho}; \boldsymbol{z})$	New extended confluent hypergeometric function.
PDF	Probability Density Function.
CDF	Cumulative Distribution Function.

1. Introduction:

The Gamma function $\Gamma(z)$ developed by Euler with the intent to extend the factorials to values between the integers is defined by the definite integral [8]:

$$\Gamma(z) = \int_0^{\infty} e^{-t}\, t^{z-1}\, dt \quad , \quad Re(z) > 0\,. \tag{1.1}$$

The Euler Beta function $B(z_1, z_2)$ (see [4, 5]) is defined by

$$B(z_1, z_2) = \int_0^{1} t^{z_1-1}\,(1-t)^{z_2-1}\, dt. \tag{1.2}$$

The classical Gauss hypergeometric functions is defined as [2, 10]:

$$_2F_1(a, c; d; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}\,, \qquad c \neq 0,1,2, \dots \tag{1.3}$$

$$|z| < -1,\;\; a, c, d \in \mathbb{C}\;;\;\; d \neq 0, -1, -2, \dots$$

and the confluent hypergeometric functions is defined by [8]:

$$_1\Phi_1(\mathrm{b}; \mathrm{c}; \mathrm{z}) = \sum_{\mathrm{n}=0}^{\infty} \frac{(\mathrm{b})_\mathrm{n}}{(\mathrm{c})_\mathrm{n}} \frac{\mathrm{z}^\mathrm{n}}{\mathrm{n}!}\,, \quad (|\mathrm{z}| < 1), \tag{1.4}$$

where $(\lambda)_n$ $(\delta \in \mathbb{C})$ is the Pochhammer symbol(see [4, 5]) is defined by

$$(\lambda)_n = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} = \begin{cases} 1 & (n=0) \\ \lambda(\lambda+1) \dots \dots \dots \dots & (\lambda+n-1) \end{cases}$$

In 1880, Appell [3], introduce four functions as follow:

$$F_1[a; b, b'; c; x, y] = \sum_{m\geq 0}^{\infty} \sum_{n\geq 0}^{\infty} \frac{(a)_{m+n}\,(b)_m\,(b')_n}{m!\,n!\,(c)_{m+n}}\, x^m y^n\,, \qquad |x|\,, |y| < 1\,, \tag{1.5}$$

$$F_2[a; b, b'; c, c'; x, y] = \sum_{m\geq 0}^{\infty} \sum_{n\geq 0}^{\infty} \frac{(a)_{m+n}\,(b)_m\,(b')_n}{m!\,n!\,(c)_m\,(c')_n} x^m y^n\,, |x| + |y| < 1\,, \tag{1.6}$$

$$F_3[a, b'; b, b'; c; x, y] = \sum_{m\geq 0}^{\infty} \sum_{n\geq 0}^{\infty} \frac{(a)_m (a')_n\,(b)_m\,(b')_n}{m!\,n!\,(c)_{m+n}} x^m y^n, |x|, |y| < 1\,, \tag{1.7}$$

$$F_4[a;b,b';c;x,y]=\sum_{m\geq 0}^{\infty}\sum_{n\geq 0}^{\infty}\frac{(a)_{m+n}\,(b)_{m+n}}{m!\,n!\,(c)_m\,(c')_n}\,x^m y^n\,,|x|^{\frac{1}{2}}+\,|y|^{\frac{1}{2}}<1\,, \qquad (1.8)$$

In (2020), Chandola et al.[4], gave the new generalized Beta functions, extended Gauss and
confluent hypergeometric functions, respectively are defined by

$${}^{F_1}B_{p,q}^{(r,r)}(x,y)=F_{p,q}^{F_1}(x,y)=\int_0^1 t^{x-1}\,(1-t)^{y-1}F_1\left(a,b,c;d;\frac{p}{t^r},\frac{q}{(1-t)^r}\right)dt. \qquad (1.9)$$

and used (1. 9) to introduce a new extended hypergeometric and confluent hypergeometric functions defined as

$${}^{F_1}F_{p,q}^{(r,r)}(\delta,\gamma;\rho;z)\,=F_{p,q}^{F_1}\,(\delta,\gamma;\rho;z)=\sum_{n=0}^{\infty}(\delta)_n\,\frac{F_{p,q}^{F_1}(\gamma+n,\rho-\gamma)}{B(\gamma,\rho-\gamma)}\,\frac{z^n}{n!}, \qquad (1.10)$$

$$\Re(\rho)>\Re(\gamma)>0\ ,\ \Re(p),\Re(q)\geq 0\,,\ |z|>-1.$$

and

$${}^{F_1}\Phi_{p,q}^{(r,r)}(\gamma;\rho;z)=\Phi_{p,q}^{F_1}(\gamma;\rho;z)=\sum_{n=0}^{\infty}\frac{\Phi_{p,q}^{F_1}(\gamma+n,\rho-\gamma)}{B(\gamma,\rho-\gamma)}\,\frac{z^n}{n!}, \qquad (1.11)$$

$$\Re(\rho)>\Re(\gamma)>0\ ,\ \Re(p),\Re(q)\geq 0\,,\ \Re(u),\Re(v)\geq 0\,,|z|<1$$

2. A new extension of beta function

Definition 2.1. The extensions of beta function using Appell series (1.5)–(1.8), respectively,
are defined as follows:

$${}^{F_1}B_{p,q}^{(u,v)}(x,y)=\int_0^1 t^{x-1}\,(1-t)^{y-1}F_1\left(a,b,b'\,;c;\frac{p}{t^u},\frac{q}{(1-t)^v}\right)dt. \qquad (2.1)$$

$$\Re(a),\Re(b),\Re(b'),\Re(c)>0\,,\Re(p)\,,\Re(q)\geq 0,\Re(u),\Re(v)\geq 0\ and\ \Re(x),\Re(y)>0$$

$${}^{F_2}B_{p,q}^{(u,v)}(x,y)=\int_0^1 t^{x-1}\,(1-t)^{y-1}F_2\left(a,b,b';c,c';\frac{p}{t^u},\frac{q}{(1-t)^v}\right)dt. \qquad (2.2)$$

$$\Re(a),\Re(b),\Re(b'),\Re(c),\Re(c')>0\,,\Re(p)\,,\Re(q)\geq 0,\Re(u),\Re(v)\geq 0\ and\ \Re(x),\Re(y)>0$$

$${}^{F_3}B_{p,q}^{(u,v)}(x,y)=\int_0^1 t^{x-1}\,(1-t)^{y-1}F_3\left(a,a',b,b';c;\frac{p}{t^u},\frac{q}{(1-t)^v}\right)dt. \qquad (2.3)$$

$$\Re(a), \Re(a'), \Re(b), \Re(b') > 0\,, \Re(p)\,, \Re(q) \geq 0, \Re(u), \Re(v) \geq 0 \; and \; \Re(x), \Re(y) > 0$$

$$^{F_4}B_{p,q}^{(u,v)}(x,y) = \int_0^1 t^{x-1}\,(1-t)^{y-1} F_4\left(a,b;c,c';\frac{p}{t^u},\frac{q}{(1-t)^v}\right)\,dt. \tag{2.4}$$

$$\Re(a), \Re(b), \Re(c), \Re(c') > 0\,, \Re(p)\,, \Re(q) \geq 0, \Re(u), \Re(v) \geq 0 \; and \; \Re(x), \Re(y) > 0$$

If $q = 0$ in {(2.1)-(2,4)}, then {(2.1)-(2,4)} reduce to the following result

$$B_p^u(x,y) = \int_0^1 t^{x-1}\,(1-t)^{y-m-1}\;{}_2F_1\left(a,c;d;\frac{p}{t^u}\right)\,dt. \tag{2.5}$$

If $p = 0$ in {(2.1)-(2,4)}, then {(2.1)-(2,4)} reduce to the following result

$$B_q^v(x,y) = \int_0^1 t^{x-n-1}\,(1-t)^{y-1}\;{}_2F_1\left(a,b;d;\frac{q}{(1-t)^v}\right)\,dt. \tag{2.6}$$

If $u = v = 1$ in (2.1)- (2.4), then (2.1)- (2.4) reduce to the following results

$$^{F_1}B_{p,q}^{(1,1)}(x,y) = B_{p,q}^{F_1}(x,y) = \int_0^1 t^{x-1}\,(1-t)^{y-1} F_1\left(a,b,b'\,;c;\frac{p}{t},\frac{q}{1-t}\right)\,dt. \tag{2.7}$$

$$^{F_2}B_{p,q}^{(1,1)}(x,y) = B_{p,q}^{F_2}(x,y) = \int_0^1 t^{x-1}\,(1-t)^{y-1} F_2\left(a,b,b';c,c';\frac{p}{t},\frac{q}{1-t}\right)\,dt. \tag{2.8}$$

$$^{F_3}B_{p,q}^{(1,1)}(x,y) = B_{p,q}^{F_3}(x,y) = \int_0^1 t^{x-1}\,(1-t)^{y-1} F_3\left(a,a',b,b';c;\frac{p}{t},\frac{q}{1-t}\right)\,dt. \tag{2.9}$$

$$^{F_4}B_{p,q}^{(1,1)}(x,y) = B_{p,q}^{F_4}(x,y) = \int_0^1 t^{x-1}\,(1-t)^{y-1} F_4\left(a,b;c,c';\frac{p}{t},\frac{q}{1-t}\right)\,dt. \tag{2.10}$$

If $u = v = r$ in (2.1)- (2.4), then (2.1)- (2.4) reduce to the results {(20)-(23)} in [4].

If $q = 0$ and $u = 1$ in {(2.1)-(2,4)}, then {(2.1)-(2,4)} reduce to the result (24) in [4].

If $p = 0$ and $v = 1$ in {(2.1)-(2,4)}, then {(2.1)-(2,4)} reduce to the following result

$$B_q(x,y) = \int_0^1 t^{x-1}\,(1-t)^{y-1}\;{}_2F_1\left(a,b;d;\frac{q}{(1-t)}\right)\,dt. \tag{2.11}$$

Subsequently, for $p = q = 0$, $a = b = c = 1$ and $u = v = 1$ equations { (2.1), (2.4)} reduce to the classical beta function $B(z_1, z_2)$, equation (1).

Theorem 2.1. The following integral representations holds true:

$$^{F_1}B_{p,q}^{(u,v)}(x,y) = 2\int_0^{\frac{\pi}{2}} \cos^{2x-1}\theta \sin^{2y-1}\theta$$
$$\times F_1(a,b,c;d;p\sec^{2u}\theta, q\, cosec^{2v}\,\theta)d\theta, \tag{2.9}$$

$$^{F_1}B_{p,q}^{(u,v)}(x,y) = \int_0^{\infty} \frac{u^{x-1}}{(1+u)^{x+y}} F_1\left(a,b,c;d;\frac{p(1+w)^u}{w}, q(1+w)^v\right)dw, \tag{2.10}$$

$$^{F_1}B_{p,q}^{(u,v)}(x,y) = 2^{1-x-y}\int_{-1}^{1}(1-w)^{x-1}(1-w)^{y-1}$$
$$\times F_1\left(a,b,c;d;\frac{2^u p}{(1+w)^u},\frac{2^v q}{(1-w)^v}\right)dw, \tag{2.11}$$

$$^{F_1}B_{p,q}^{(u,v)}(x,y) = (c-z)^{1-x-y}\int_z^c (w-z)^{x-1}(c-w)^{y-1}$$
$$\times F_1\left(a,b,c;d;p\left(\frac{c-z}{w-z}\right)^u, q\left(\frac{c-z}{c-w}\right)^v\right)dw, \tag{2.12}$$

$$^{F_1}B_{p,q}^{(u,v)}(x,y) = 2\int_0^{\frac{\pi}{2}} \tanh^{2x-1}\theta\, \mathrm{sech}^{2y-1}\theta$$
$$\times F_1(a,b,c;d;p\coth^{2u}\theta, q\, cosh^{2v}\,\theta)d\theta, \tag{2.13}$$

Proof: To prove the formula (2.9), putting $t = \cos^2\theta$ in (2.1), we have

$$^{F_1}B_{p,q}^{(u,v)}(x,y) = 2\int_0^{\frac{\pi}{2}} (\cos^2\theta)^{x-1}(1-\cos^2\theta)^{y-1}$$
$$F_1\left(a,b,c;d;\frac{p}{\cos^{2u}\theta},\frac{q}{1-\cos^{2v}\theta}\right)\cos\theta\sin\theta\, d\theta$$

$$^{F_1}B_{p,q}^{(u,v)}(x,y) = 2\int_0^{\frac{\pi}{2}} \cos^{2x-1}\theta\sin^{2y-1}\theta\; F_1\left(a,b,c;d;\frac{p}{\cos^{2u}\theta},\frac{q}{\sin^{2v}\theta}\right)d\theta$$

$${}^{F_1}B_{p,q}^{(u,v)}(x,y) = 2\int_0^{\frac{\pi}{2}} \cos^{2x-1}\theta \sin^{2y-1}\theta \ F_1(a,b,c;d;p\sec^2\theta, q\,cosec^2\ \theta)d\theta$$

Similarly, the formulas (3.10) to (2.13) can be proved by taking the transformation, $t=\frac{w}{1+w}$ and $t=\frac{1+w}{2}$, $t=\frac{(w-z)}{(c-w)}$ and $t=\tanh^2\theta$ in (2.1), respectively.

Similarly, we can prove the above results for ${}^{F_2}B_{p,q}^{(u,v)}(x,y)$, ${}^{F_3}B_{p,q}^{(u,v)}(x,y)$, ${}^{F_4}B_{p,q}^{(u,v)}(x,y)$

Remark 2.1. If $q=0$ in $\{(2.9)-(2.13)\}$, then $\{(2.9)-(2.13)\}$ reduce to the following results

$${}^{{}_2F_1}B_p^u(x,y) = 2\int_0^{\frac{\pi}{2}} \cos^{2x-1}\theta \sin^{2y-1}\theta \ \ {}_2F_1(a,c;d;p\sec^2\theta,)d\theta, \tag{2.14}$$

$${}^{{}_2F_1}B_{p,q}^u(x,y) = \frac{u^{x-1}}{(1+u)^{x+y}} \ {}_2F_1\left(a,c;d;\frac{p(1+u)}{u},\right)du, \tag{2.15}$$

$${}^{{}_2F_1}B_p^u(x,y) = 2^{1-x-y}\int_{-1}^{1}(1-u)^{x-1}(1-u)^{y-1} \times \ {}_2F_1\left(a,c;d;\frac{2p}{1+u}\right)du\,, \tag{2.16}$$

$${}^{{}_2F_1}B_p^u(x,y) = (c-z)^{1-x-y}\int_z^c (w-z)^{x-1}(c-w)^{y-1}$$

$$\times \ {}_2F_1\left(a,b,c;d;p\left(\frac{c-z}{w-z}\right)^u\right)dw\,, \tag{2.17}$$

$${}^{{}_2F_1}B_p^u(x,y) = 2\int_0^{\frac{\pi}{2}} \tanh^{2x-1}\theta\,\mathrm{sech}^{2y-1}\theta \times \ {}_2F_1(a,b,c;d;p\coth^{2u}\theta,)d\theta, \tag{2.18}$$

If $p=0$ in $\{(2.9)-(2.13)\}$, then $\{(2.9)-(2.13)\}$ reduce to the following results

$${}^{{}_2F_1}B_q^v(x,y) \ = 2\int_0^{\frac{\pi}{2}} \cos^{2x-1}\theta \sin^{2y-1}\theta \ \ {}_2F_1(a,b;d;q\,cosec^2\ \theta)d\theta, \tag{2.19}$$

$${}^{{}_2F_1}B_q^v(x,y) \ = \frac{u^{x-1}}{(1+u)^{x+y}} \ \ {}_2F_1\big(a,b;d;q(1+u)\big)du, \tag{2.20}$$

$${}^{{}_2F_1}B_q^v(x,y) = 2^{1-x-y}\int_{-1}^{1}(1-u)^{x-1}(1-u)^{y-1} \ \ {}_2F_1\left(a,b;d;\frac{2q}{1-u}\right)du\,, \tag{2.21}$$

$$ {}_{2}^{F_1}B_q^{v}(x,y) = (c-z)^{1-x-y}\int_z^c (w-z)^{x-1}(c-w)^{y-1} $$

$$ \times\ {}_2F_1\left(a,b,c;d;q\left(\frac{c-z}{c-w}\right)^{v}\right)dw\,, \qquad (2.22) $$

$$ {}_{2}^{F_1}B_q^{v}(x,y) = 2\int_0^{\frac{\pi}{2}} \tanh^{2x-1}\theta\,\operatorname{sech}^{2y-1}\theta\ \ {}_2F_1(a,b,c;d;q\,cosh^{2v}\ \theta)d\theta, \qquad (2.23) $$

If $u=v=r$ in $\{(2.9)-(2.13)\}$, then $\{(2.9)-(2.13)\}$ reduce to the results {(34) - (38)} in [3].

The study of special function extensions is not merely an abstract mathematical exercise but a pursuit with significant implications for applied mathematics and theoretical physics. The new extensions proposed in this work, ${}^{F_i}B_{p,q}^{(u,v)}(x,y)$ for $i=1,2,3,4$, hold potential for applications in developing complex statistical distributions for modeling real-world phenomena with heavy-tailed behavior [1, 9], in defining novel fractional integral operators for solving advanced problems in fractional calculus [4, 7, 12], and in finding solutions to partial differential equations encountered in quantum mechanics and transport theory. This paper aims to provide a robust theoretical foundation for these functions, upon which future applied work can be built

3. properties of extended beta function

Theorem 3.1. The extension of beta function satisfies the following integral representation

$$ {}^{F_1}B_{p,q}^{(u,v)}(x+1,y) + {}^{F_1}B_{p,q}^{(a,b,c;d)}(x,y+1) = {}^{F_1}B_{p,q}^{(a,b,c;d)}(x,y). \qquad (3.1) $$

$$ {}^{F_2}B_{p,q}^{(u,v)}(x+1,y) + {}^{F_2}B_{p,q}^{(a,b,c;d)}(x,y+1) = {}^{F_2}B_{p,q}^{(a,b,c;d)}(x,y). \qquad (3.2) $$

$$ {}^{F_3}B_{p,q}^{(u,v)}(x+1,y) + {}^{F_3}B_{p,q}^{(a,b,c;d)}(x,y+1) = {}^{F_3}B_{p,q}^{(a,b,c;d)}(x,y). \qquad (3.3) $$

$$ {}^{F_4}B_{p,q}^{(u,v)}(x+1,y) + {}^{F_4}B_{p,q}^{(a,b,c;d)}(x,y+1) = {}^{F_4}B_{p,q}^{(a,b,c;d)}(x,y). \qquad (3.4) $$

Proof. Consider the left-hand side of (3.1), we have

$$ {}^{F_1}B_{p,q}^{(u,v)}(x+1,y) + {}^{F_1}B_{p,q}^{(u,v)}(x,y+1) $$

$$= \int_0^1 \{t^x(1-t)^{y-1} + t^{x-1}(1-t)^y\} F_1\left(a,b,c;d;\frac{p}{t^u},\frac{q}{(1-t)^v}\right) dt$$

$${}^{F_1}B_{p,q}^{(u,v)}(x+1,y) + {}^{F_1}B_{p,q}^{(u,v)}(x,y+1)$$

$$= \int_0^1 \{t^{x-1}(1-t)^{y-1}(t+1-t)\} F_1\left(a,b,c;d;\frac{p}{t^u},\frac{q}{(1-t)^v}\right) dt$$

$${}^{F_1}B_{p,q}^{(u,v)}(x+1,y) + {}^{F_1}B_{p,q}^{(u,v)}(x,y+1)$$

$$= \int_0^1 t^{x-1}(1-t)^{y-1} F_1\left(a,b,c;d;\frac{p}{t^u},\frac{q}{(1-t)^v}\right) dt$$

$${}^{F_1}B_{p,q}^{(u,v)}(x+1,y) + {}^{F_1}B_{p,q}^{(u,v)}(x,y+1) = {}^{F_1}B_{p,q}^{(u,v)}(x,y)$$

Similarly, we can prove the results (3.2), (3.3) and (3.4).

The recurrence relations established in Theorem 3.1 generalize the fundamental property $B(x+1,y) + B(x,y+1) = B(x,y)$ of the classical Beta function [8]. These relations are not merely algebraic curiosities; they are crucial for the computational evaluation of the extended Beta functions, allowing for the development of efficient recursive algorithms, especially when dealing with large parameter values, similar to those employed for other special functions [6].

Theorem 3.2. Let $Re(x) > 0$, $Re(y) < 1$, $Re(p) > 0$, $Re(\alpha) > -1$. Then,

$${}^{F_1}B_{p,q}^{(u,v)}(x,1-y) = \sum_0^\infty \frac{(y)_n}{n!}\ {}^{F_1}B_{p,q}^{(u,v)}(x+n,1). \tag{3.5}$$

$${}^{F_2}B_{p,q}^{(u,v)}(x,1-y) = \sum_0^\infty \frac{(y)_n}{n!}\ {}^{F_2}B_{p,q}^{(u,v)}(x+n,1). \tag{3.6}$$

$${}^{F_3}B_{p,q}^{(u,v)}(x,1-y) = \sum_0^\infty \frac{(y)_n}{n!}\ {}^{F_3}B_{p,q}^{(u,v)}(x+n,1). \tag{3.7}$$

$${}^{F_4}B_{p,q}^{(u,v)}(x,1-y) = \sum_0^\infty \frac{(y)_n}{n!}\ {}^{F_4}B_{p,q}^{(u,v)}(x+n,1). \tag{3.8}$$

Proof: To prove (3.5), from (2.1), we have

$$^{F_1}B_{p,q}^{(u,v)}(x,1-y) = \int_0^1 t^{x-1}\,(1-t)^{-y} F_1\left(a,b,c;d;\frac{p}{t^u},\frac{q}{(1-t)^v}\right)dt,$$

using the generalized binomial theorem

$$(1-t)^{-y} = \sum_{n=0}^{\infty}\frac{(y)_n}{n!}t^n, \quad |t|<1\,,$$

We obtain

$$^{F_1}B_{p,q}^{(u,v)}(x,1-y) = \sum_{n=0}^{\infty}\frac{(y)_n}{n!}t^{x+n-1}F_1\left(a,b,c;d;\frac{p}{t^u},\frac{q}{(1-t)^v}\right)dt.$$

Now, interchanging the order of summation and integration in above equation and using

(2.1) proves the desired result.

Similarly, we can prove the results (3.6), (3.7) and (3.8).

Theorem 3.3. The extension of beta function satisfies the following infinite summation
Formulas:

$$^{F_1}B_{p,q}^{(u,v)}(x,y) = \sum_{n=0}^{\infty} {}^{F_1}B_{p,q}^{(u,v)}(x+n,y+1). \tag{3.9}$$

$$^{F_2}B_{p,q}^{(u,v)}(x,y) = \sum_{n=0}^{\infty} {}^{F_2}B_{p,q}^{(u,v)}(x+n,y+1). \tag{3.10}$$

$$^{F_3}B_{p,q}^{(u,v)}(x,y) = \sum_{n=0}^{\infty} {}^{F_3}B_{p,q}^{(u,v)}(x+n,y+1). \tag{3.11}$$

$$^{F_4}B_{p,q}^{(u,v)}(x,y) = \sum_{n=0}^{\infty} {}^{F_4}B_{p,q}^{(u,v)}(x+n,y+1). \tag{3.12}$$

Proof. To prove the result (3.9), replacing the following series representation in (2.1)

$$(1-t)^{y-1} = (1-t)^y\sum_{n=0}^{\infty}t^n \qquad (|t|<1),$$

we obtain

$$^{F_1}B_{p,q}^{(u,v)}(x,y) = \int_0^1 (1-t)^y\sum_{n=0}^{\infty}t^{n+x-1}\;\; F_1\left(a,b,c;d;\frac{p}{t^u},\frac{q}{(1-t)^v}\right)dt\,,$$

by interchanging the order of integration and summation in above equation and using (2.1), we get the desired result.

Similarly, we can prove the results (3.10), (3.11) and (3.12).

Theorem 3.4. The following relation holds true

$$ {}^{F_1}B_{p,q}^{(u,v)}(x,y) = \sum_{k=0}^{\infty} \binom{n}{k} \, {}^{F_1}B_{p,q}^{(u,v)}(x+k, y+n-k). \quad n \in \mathbb{N}_0 \tag{3.13} $$

$$ {}^{F_2}B_{p,q}^{(u,v)}(x,y) = \sum_{k=0}^{\infty} \binom{n}{k} \, {}^{F_2}B_{p,q}^{(u,v)}(x+k, y+n-k). \quad n \in \mathbb{N}_0 \tag{3.12} $$

$$ {}^{F_3}B_{p,q}^{(u,v)}(x,y) = \sum_{k=0}^{\infty} \binom{n}{k} \, {}^{F_3}B_{p,q}^{(u,v)}(x+k, y+n-k). \quad n \in \mathbb{N}_0 \tag{3.13} $$

$$ {}^{F_4}B_{p,q}^{(u,v)}(x,y) = \sum_{k=0}^{\infty} \binom{n}{k} \, {}^{F_4}B_{p,q}^{(u,v)}(x+k, y+n-k). \quad n \in \mathbb{N}_0 \tag{3.14} $$

Proof. To prove the result (3,13), we find from (2.1) that

$$ {}^{F_1}B_{p,q}^{(u,v)}(x,y) = \int_0^1 t^{x-1} (1-t)^{y-1} [t + (1-t)] F_1\left(a,b,c;d;\frac{p}{t^u},\frac{q}{(1-t)^v}\right) dt, $$

$$ = {}^{F_1}B_{p,q}^{(u,v)}(x+1,y) + {}^{F_1}B_{p,q}^{(u,v)}(x,y+1) \tag{3.15} $$

Repeating the same argument to the above two terms in (3.23), we obtain

$$ {}^{F_1}B_{p,q}^{(u,v)}(x,y) = {}^{F_1}B_{p,q}^{(u,v)}(x+2,y) + 2\,{}^{F_1}B_{p,q}^{(u,v)}(x+1,y+1)\,{}^{F_1}B_{p,q}^{(u,v)}(x,y+2). \tag{3.16} $$

Continuing this process, by using mathematical induction we get the desired result (3.19).

Similarly, we can prove the results (3.12), (3.13) and (3.14).

4. Statistical distribution involving extended beta function

In this section, application of the newly defined extension of beta function in statistics has been discussed. We define the extended beta distribution and derive the results for its mean, variance, moment generating function and cumulative distribution function.

Definition 4.1. Distribution of a new extended beta function involving Appell function $F_1(.)$ is defined by

$$f(t) = \begin{cases} \frac{1}{{}^{F_1}B_{p,q}^{(u,v)}(x,y)} \; t^{x-1}\,(1-t)^{y-1}\; F_1\left(a,b,b';c;\frac{p}{t^u},\frac{q}{(1-t)^v}\right), (0<t<1), \\ 0, \qquad\qquad otherwise \end{cases} \tag{4.1}$$

$$\Re(a),\Re(b),\Re(b'),\Re(c)>0\,,\Re(p)\,,\Re(q)\geq 0,\Re(u),\Re(v)\geq 0 \;and\; \Re(x),\Re(y)>0$$

The proposed extended Beta distribution provides greater flexibility for modeling real-world data compared to the classical Beta distribution, similar to other recent extensions [1, 9]. The introduction of additional parameters $(p,q,u,v,a,b,\mathrm{b}',\mathrm{c})$ through the Appell function F_1 allows for a significantly more complex shape, accommodating higher skewness, multi-modality, and heavier tails. This makes it a more suitable candidate for fitting diverse and complex datasets in various fields such as finance, biology, and engineering.

The statistical properties derived here (mean, variance, MGF, CDF) follow the standard methodology for distributions defined through special functions [4]. This methodological consistency ensures the theoretical soundness of our results and allows for direct comparison with the properties of other well-established distributions.

For $d \in R$, the d^{th} moment of a random variable X defined as

$$\rho = E(X^d) = \frac{{}^{F_1}B_{p,q}^{(u,v)}(x+d,\; y)}{{}^{F_1}B_{p,q}^{(u,v)}(x,y)}, \tag{4.2}$$

The variance of the distribution is defined by

$$\sigma^2 = E(X^2) - \left(E(X)\right)^2 = \frac{{}^{F_1}B_{p,q}^{(u,v)}(x,\; y) + {}^{F_1}B_{p,q}^{(u,v)}(x+2,y) - \left\{{}^{F_1}B_{p,q}^{(u,v)}(x+1,y)\right\}^2}{\left\{{}^{F_1}B_{p,q}^{(u,v)}(x,y)\right\}^2}. \tag{4.3}$$

The moment generating function of the distribution is defined as

$$M(t)=\sum_{n=0}^{\infty}\frac{t^n}{n!}\,E(X^n)=\frac{1}{{}^{F_1}B_{p,q}^{(u,v)}(x,y)}\sum_{n=0}^{\infty}{}^{F_1}B_{p,q}^{(u,v)}(x+n,y)\,\frac{t^n}{n!}. \tag{4.4}$$

The cumulative distribution is defined as

$$f(z)=\frac{{}^{F_1}B_{p,q}^{(u,v)}(x+d,\ y)}{{}^{F_1}B_{p,q}^{(u,v)}(x,y)}. \tag{4.5}$$

where

$$ {}^{F_1}B_{p,q}^{(u,v)}(x,y)=\int_0^z t^{x-1}\,(1-t)^{y-1}F_1\left(a,b,b'\,;c;\frac{p}{t^u},\frac{q}{(1-t)^v}\right)dt, \tag{4.6}$$

is the extended incomplete Beta function.

Similarly, we can prove the above results for ${}^{F_2}B_{p,q}^{(u,v)}(x,y)$, ${}^{F_3}B_{p,q}^{(u,v)}(x,y)$, ${}^{F_4}B_{p,q}^{(u,v)}(x,y)$.

5. New Generalized Gauss and Confluent Hypergeometric Functions

In this section, we introduced new generalized Gauss and confluent hypergeometric functions and presented some of their properties.

Definition 5.1. The extension of hypergeometric function using the newly defined beta
function involving Appell series are

$$ {}^{F_1}F_{p,q}^{(u,v)}(\delta,\gamma;\rho;z)=\sum_{n=0}^{\infty}(\delta)_n\,\frac{{}^{F_1}B_{p,q}^{(u,v)}(\gamma+n,\rho-\gamma)}{B(\gamma,\rho-\gamma)}\,\frac{z^n}{n!}, \tag{5.1}$$

$$\Re(\rho)>\Re(\gamma)>0\ ,\ \Re(p),\Re(q)\geq 0\ ,\ \Re(u),\Re(v)\geq 0\ |z|<1,$$

$$ {}^{F_2}F_{p,q}^{(u,v)}(\delta,\gamma;\rho;z)=\sum_{n=0}^{\infty}(\delta)_n\,\frac{{}^{F_2}B_{p,q}^{(u,v)}(\gamma+n,\rho-\gamma)}{B(\gamma,\rho-\gamma)}\,\frac{z^n}{n!}, \tag{5.2}$$

$\Re(\rho)>\Re(\gamma)>0\ ,\ \Re(p),\Re(q)\geq 0\ ,\ \Re(u),\Re(v)\geq 0\ ,\ |z|<1$

$$ {}^{F_3}F_{p,q}^{(u,v)}(\delta,\gamma;\rho;z)=\sum_{n=0}^{\infty}(\delta)_n\,\frac{{}^{F_3}B_{p,q}^{(u,v)}(\gamma+n,\rho-\gamma)}{B(\gamma,\rho-\gamma)}\,\frac{z^n}{n!}, \tag{5.3}$$

$\Re(\rho)>\Re(\gamma)>0\ ,\ \Re(p),\Re(q)\geq 0\ ,\ \Re(u),\Re(v)\geq 0\ ,\ |z|<1$

$$ {}^{F_4}F_{p,q}^{(u,v)}(\delta,\gamma;\rho;z)=\sum_{n=0}^{\infty}(\delta)_n\,\frac{{}^{F_4}B_{p,q}^{(u,v)}(\gamma+n,\rho-\gamma)}{B(\gamma,\rho-\gamma)}\,\frac{z^n}{n!}, \tag{5.4}$$

$\Re(\rho) > \Re(\gamma) > 0$, $\Re(p), \Re(q) \geq 0$, $\Re(u), \Re(v) \geq 0$, $|z| < 1$

Definition 5.2. The extension of confluent hypergeometric function using the newly defined beta function involving Appell series are

$$^{F_1}\Phi_{p,q}^{(u,v)}(\gamma;\rho;z) = \sum_{n=0}^{\infty} \frac{^{F_1}\mathrm{B}_{p,q}^{(u,v)}(\gamma+n,\rho-\gamma)}{B(\gamma,\rho-\gamma)} \frac{z^n}{n!}, \tag{5.5}$$

$\Re(\rho) > \Re(\gamma) > 0$, $\Re(p), \Re(q) \geq 0$, $\Re(u), \Re(v) \geq 0$, $|z| < 1$

$$^{F_2}\Phi_{p,q}^{(u,v)}(\gamma;\rho;z) = \sum_{n=0}^{\infty} \frac{^{F_2}\mathrm{B}_{p,q}^{(u,v)}(\gamma+n,\rho-\gamma)}{B(\gamma,\rho-\gamma)} \frac{z^n}{n!}, \tag{5.6}$$

$\Re(\rho) > \Re(\gamma) > 0$, $\Re(p), \Re(q) \geq 0$, $\Re(u), \Re(v) \geq 0$, $|z| < 1$

$$^{F_3}\Phi_{p,q}^{(u,v)}(\gamma;\rho;z) = \sum_{n=0}^{\infty} \frac{^{F_3}\mathrm{B}_{p,q}^{(u,v)}(\gamma+n,\rho-\gamma)}{B(\gamma,\rho-\gamma)} \frac{z^n}{n!}, \tag{5.7}$$

$\Re(\rho) > \Re(\gamma) > 0$, $\Re(p), \Re(q) \geq 0$, $\Re(u), \Re(v) \geq 0$, $\Re(\alpha) > -1$

$$^{F_4}\Phi_{p,q}^{(u,v)}(\gamma;\rho;z) = \sum_{n=0}^{\infty} \frac{^{F_4}\mathrm{B}_{p,q}^{(u,v)}(\gamma+n,\rho-\gamma)}{B(\gamma,\rho-\gamma)} \frac{z^n}{n!}, \tag{5.8}$$

$\Re(\rho) > \Re(\gamma) > 0$, $\Re(p), \Re(q) \geq 0$, $\Re(u), \Re(v) \geq 0$, $|z| < 1$

Our definition of extended hypergeometric functions generalizes the approach taken by earlier researchers who extended these functions using other special functions [5, 6, 11]. By incorporating the Appell functions into the kernel, we introduce a higher level of generality and flexibility, which allows for a wider range of applications in mathematical physics and fractional calculus.

Remark 5.1. When $u = v = r$ in (5.1) and (5.5), then (5.1) and (5.5) reduce to the results (44) and (45) in [3].

Remark 5.2. When $q = 0$ and subsequently if $p = 1$, $u = v = 0$, $\delta, \gamma, \rho = 1$, equations (5.1)–(5.8) reduce to the Gauss hypergeometric function (1.3) and confluent hypergeometric function (1.4), respectively

6. Integral representations of extended hypergeometric functions

In this section, we derive differentiation formulas for the extended hypergeometric and confluent hypergeometric functions.

The integral representations derived here generalize those known for classical hypergeometric functions [2, 10] and for other extended functions [4, 6].

Theorem 6.1. The extended hypergeometric has the following integral representations.

$$ {}^{F_1}F_{p,q}^{(u,v)}(\delta,\gamma;\rho;z) = \frac{1}{B(\gamma,\rho-\gamma)} \times \int_0^1 t^{\gamma-1}\,(1-t)^{\rho-\gamma-1}(1-zt)^{-\delta}\,F_1\left(a,b,c;d;\frac{p}{t^u},\frac{q}{(1-t)^v}\right)dt. \tag{6.1} $$

$$ {}^{F_2}F_{p,q}^{(u,v)}(\delta,\gamma;\rho;z) = \frac{1}{B(\gamma,\rho-\gamma)} \times \int_0^1 t^{\gamma-1}\,(1-t)^{\rho-\gamma-1}(1-zt)^{-\delta}\,F_2\left(a,b,c;d;\frac{p}{t^u},\frac{q}{(1-t)^v}\right)dt. \tag{6.2} $$

$$ {}^{F_3}F_{p,q}^{(u,v)}(\delta,\gamma;\rho;z) = \frac{1}{B(\gamma,\rho-\gamma)} \times \int_0^1 t^{\gamma-1}\,(1-t)^{\rho-\gamma-1}(1-zt)^{-\delta}\,F_3\left(a,b,c;d;\frac{p}{t^u},\frac{q}{(1-t)^v}\right)dt. \tag{6.3} $$

$$ {}^{F_4}F_{p,q}^{(u,v)}(\delta,\gamma;\rho;z) = \frac{1}{B(\gamma,\rho-\gamma)} \times \int_0^1 t^{\gamma-1}\,(1-t)^{\rho-\gamma-1}(1-zt)^{-\delta}\,F_4\left(a,b,c;d;\frac{p}{t^u},\frac{q}{(1-t)^v}\right)dt. \tag{6.4} $$

Proof. To prove (6.1), we are using (2.1) in (5.1), we have

$$ {}^{F_1}F_{p,q}^{(u,v)}(\delta,\gamma;\rho;z) = \frac{1}{B(\gamma,\rho-\gamma)} \sum_{n=0}^{\infty}(\delta)_n \int_0^1 t^{\gamma+n-1}\,(1-t)^{\rho-\gamma-1}\,F_1\left(a,b,c;d;\frac{p}{t^u},\frac{q}{(1-t)^v}\right)dt\,\frac{z^n}{n!}, $$

$$ {}^{F_1}F_{p,q}^{(u,v)}(\delta,\gamma;\rho;z) = \frac{1}{B(\gamma,\rho-\gamma)} $$

$$\int_0^1 t^{\gamma-1}\,(1-t)^{\rho-\gamma-1}\,F_1\left(a,b,c;d;\frac{p}{t^u},\frac{q}{(1-t)^v}\right)dt\,\sum_{n=0}^{\infty}(\delta)_n\,\frac{(zt)^n}{n!},$$

$$^{F_1}F_{p,q}^{(u,v)}(\delta,\gamma;\rho;z)=\frac{1}{B(\gamma,\rho-\gamma)}\int_0^1 t^{\gamma-1}\,(1-t)^{\rho-\gamma-1}(1-zt)^{-\delta}\,F_1\left(a,b,c;d;\frac{p}{t^u},\frac{q}{(1-t)^v}\right)dt$$

Similarly, we can prove the results (6.2), (6.3) and (6.4).

Theorem 6.2. The following integral representations holds true:

$$^{F_1}F_{p,q}^{(u,v)}(\delta,\gamma;\rho;z)=\frac{2}{B(\gamma,\rho-\gamma)}\int_0^{\frac{\pi}{2}}(\sin\theta)^{2\gamma-1}(\cos\theta)^{2\rho-2\gamma-1}$$

$$(1-z\sin^2\theta)^{-\delta}\,F_1(a,b,c;d;p\sec^{2u}\theta\,,q\csc^{2v}\theta)\,d\theta \qquad (6.5)$$

$$^{F_1}F_{p,q}^{(u,v)}(\delta,\gamma;\rho;z)=\frac{1}{B(\gamma,\rho-\gamma)}\int_0^{\infty} w^{\gamma-1}\,(1+w)^{\delta-\rho}\big(1+u(1-z)\big)^{-\delta}$$

$$\times F_1\left(a,b,c;d;\frac{p(1+w)^u}{w^u},q(1+w)^v\right)dw\,, \qquad (6.6)$$

$$^{F_1}F_{p,q}^{(u,v)}(\delta,\gamma;\rho;z)=\frac{2^{1+\delta-\rho}}{B(\gamma,\rho-\gamma)}\int_{-1}^{1}(1-w)^{\rho-\gamma-1}\,\big(2-z(1+w)\big)^{-\delta}\,(1+w)^{\gamma-1}$$

$$F_1\left(a,b,c;d;\,\frac{2^u p}{(1+w)^u},\frac{2^v q}{(1-w)^v}\right)\,dw. \qquad (6.7)$$

$$^{F_1}F_{p,q}^{(u,v)}(\delta,\gamma;\rho;z)=\frac{(c-z)^{1+\delta-\rho}}{B(\gamma,\rho-\gamma)}\int_w^c(a-w)^{\gamma-1}\,[(c-w)-z(a-w)]^{-\delta}\,(c-w)^{\rho-\gamma-1}$$

$$\times F_1\left(a,b,c;d;p\left(\frac{c-w}{a-w}\right)^u,q\left(\frac{c-w}{c-a}\right)^v\right)dw\,, \qquad (6.8)$$

$$^{F_1}F_{p,q}^{(u,v)}(\delta,\gamma;\rho;z)=\frac{2}{B(\gamma,\rho-\gamma)}\int_0^{\frac{\pi}{4}}(\mathrm{tanhh}\,\theta)^{2\gamma-2}(\mathrm{sech}\,\theta)^{2\rho-2\gamma}\;\;(1-z\tanh^2\theta)^{-\delta}$$

$$F_1(a,b,c;d;p\coth^{2u}\theta\,,q\cosh^{2v}\theta)\,d\theta \qquad (6.9)$$

Prove: Using (2.9) -(2.13) in (1.4), respectively, we get the desired results (6.5) -(5.9).

Theorem 6.3. The following integral representations for the extended confluent hypergeometric
function hold true:

$$ {}^{F_1}\Phi_{p,q}^{(u,v)}(\gamma;\rho;z) = \frac{2}{B(\gamma,\rho-\gamma)} \int_0^{\frac{\pi}{2}} (\sin\theta)^{2\gamma-1}(\cos\theta)^{2\rho-2\gamma-1}\ exp(z\cos^2\theta) $$

$$ F_1(a,b,c;d;p\sec^{2u}\theta, q\csc^{2v}\theta)\, d\theta \tag{6.10} $$

$$ {}^{F_1}\Phi_{p,q}^{(u,v)}(\gamma;\rho;z) = \frac{1}{B(\gamma,\rho-\gamma)} $$

$$ \times \int_0^\infty w^{\gamma-1}(1+w)^{-\rho} \exp\left(\frac{wz}{1+w}\right) F_1\left(a,b,c;d;\frac{p\,(1+w)^u}{t^u}, q(1-t)^v\right) dt, \tag{6.11} $$

$$ {}^{F_1}\Phi_{p,q}^{(u,v)}(\gamma;\rho;z) = \frac{2^{1-\rho}}{B(\gamma,\rho-\gamma)} \int_{-1}^{1} (1+w)^{\gamma-1}\ (1-w)^{\rho-\gamma-1}\ \exp\left(\frac{z(1+w)}{2}\right) $$

$$ \times F_1\left(a,b,c;d;\ \frac{2^u p}{(1+w)^u}, \frac{2^v q}{(1-w)^v}\right) dw. \tag{6.12} $$

$$ {}^{F_1}\Phi_{p,q}^{(u,v)}(\gamma;\rho;z) = \frac{(c-w)^{1-\rho}}{B(\gamma,\rho-\gamma)} \int_w^c (a-w)^{\gamma-1}(c-a)^{\rho-\gamma-1} \exp\left(\frac{z(a-w)}{(c-w)}\right) $$

$$ \times F_1\left(a,b,c;d;p\left(\frac{c-w}{a-w}\right)^u, q\left(\frac{c-w}{c-a}\right)^v\right) da, \tag{6.13} $$

$$ {}^{F_1}\Phi_{p,q}^{(u,v)}(\gamma;\rho;z) = \frac{2}{B(\gamma,\rho-\gamma)} \int_0^{\frac{\pi}{4}} (\tanh\theta)^{2\gamma-2}(\operatorname{sech}\theta)^{2\rho-2\gamma}\ exp(z\tanh^2\theta) $$

$$ \times F_1(a,b,c;d;p\coth^{2u}\theta, q\cosh^{2v}\theta)\, d\theta \tag{6.14} $$

Prove: Using (2.9)-(2.13) in (4.5), respectively, we get the desired results (6.10)-(6.14).

7. Differentiation formulas

In this section, we derive differentiations formulas for the extended hypergeometric and confluent hypergeometric functions.

The differentiation formulas presented in Theorem 7.1 follow the same pattern as those for classical hypergeometric functions [2, 8] and their various extensions [5, 11]. This consistency underscores the natural generalization achieved by our new definitions.

Theorem 7.1. The following formula holds true:

$$\frac{d}{dz}\left\{{}^{F_1}F_{p,q}^{(u,v)}(\delta,\gamma;\rho;z)\right\} = \frac{(\delta)_n\,(\gamma)_n}{(\rho)_n} F_1\left(\delta+n,\gamma+n;\ \rho+n;\frac{p}{t^u},\frac{q}{(1-t)^v}\right). \tag{7.1}$$

Proof. Differentiating (4.1) with respect to z , we have

$$\frac{d}{dz}\left\{{}^{F_1}F_{p,q}^{(u,v)}(\delta,\gamma;\rho;z)\right\} = \frac{d}{dz}\left(\sum_{n=0}^{\infty}(\delta)_n\,\frac{{}^{F_1}F_{p,q}^{(u,v)}(\gamma+n,\rho-\gamma)}{B(\gamma,c-\gamma)}\;\frac{z^n}{n!}\right),$$

$$= \sum_{n=1}^{\infty}(\delta)_n\,\frac{{}^{F_1}F_{p,q}^{(u,v)}(\gamma+n,\rho-\gamma)}{B(\gamma,c-\gamma)}\;\frac{z^{n-1}}{(n-1)!}, \tag{7.2}$$

changing n to $n+1$ in (7.2), we have

$$\frac{d}{dz}\left\{{}^{F_1}F_{p,q}^{(u,v)}(\delta,\gamma;\rho;z)\right\} = \sum_{n=1}^{\infty}(\delta)_{n+1}\,\frac{{}^{F_1}F_{p,q}^{(u,v)}(\gamma+n,\rho-\gamma)}{B(\gamma,c-\gamma)}\;\frac{z^n}{n!}, \tag{7.3}$$

since

$$B(b,c-b) = \tfrac{c}{b}\,B(b+1,c-b). \tag{7.4}$$

Applying (7.4) in (7.3), we get

$$\frac{d}{dz}\left\{{}^{F_1}F_{p,q}^{(u,v)}(\delta,\gamma;\rho;z)\right\} = \frac{\delta\,\gamma}{\rho}\sum_{n=1}^{\infty}(\delta+1)_n\,\frac{{}^{F_1}F_{p,q}^{(u,v)}(\gamma+n,\rho-\gamma)}{B(\gamma+1,\rho-\gamma)}\;\frac{z^n}{n!}$$

$$= \frac{\delta\,\gamma}{\rho}\,{}^{F_1}F_{p,q}^{(u,v)}(\delta+1,\gamma+1;\rho+1;z), \tag{7.5}$$

again differentiating (7.5) with respect to z, we obtain

$$\frac{d^2}{dz^2}\left\{{}^{F_1}F_{p,q}^{(u,v)}(\delta,\gamma;\rho;z)\right\} = \frac{(\delta+1)\,(\gamma+1)}{(\rho+1)}\,{}^{F_1}F_{p,q}^{(u,v)}(\delta+2,\gamma+2;\rho+2;z), \tag{7.6}$$

continuing up to n times, we get the required result.

Theorem 7.2. The following formula hold true:

$$\frac{d^n}{dz^n}\left\{{}^{F_1}\Phi_{p,q}^{(u,v)}(\gamma;\rho;z)\right\} = \frac{(\gamma)_n}{(\rho)_n}\,{}^{F_1}\Phi_{p,q}^{(u,v)}(\gamma+n;\rho+n;z). \tag{7.7}$$

Proof. Applying the similar procedure used in Theorem 7.1, we get the desired result.

Similarly, we can prove the above results for ${}^{F_2}F_{p,q}^{(u,v)}(\delta,\gamma;\rho;z)$, ${}^{F_3}F_{p,q}^{(u,v)}(\delta,\gamma;\rho;z)$, ${}^{F_4}F_{p,q}^{(u,v)}(\delta,\gamma;\rho;z)$.

8. Transformation and summation formulas

In this section, we obtain transformation and summation formulas for the extended hypergeomet- ric and confluent hypergeometric functions as follows:
The transformation formulas in Theorem 8.1 generalize the well-known transformations of Gauss hypergeometric functions [2, 10] to the case of extended functions with Appell series in their kernel. This provides a powerful tool for simplifying complex expressions involving the new functions.
The summation formula in Theorem 8.2 extends similar results for classical special functions [8] and their recent generalizations [4, 6]. This further demonstrates the unifying nature of the proposed extension

Theorem 8.1. The following transformation for extended hypergeometric and confluent hypergeometric functions holds true for $p,q \geq 0$ *and* $\alpha,\beta \in \Re^+$:

$${}^{F_1}F_{p,q}^{(u,v)}(\delta,\gamma;\rho;z) = (1-z)^{-\delta}\,{}^{F_1}F_{q,p}^{(u,v)}\left(\delta,\gamma;\rho;\frac{z}{1-z}\right), \tag{8.1}$$

$${}^{F_1}F_{p,q}^{(u,v)}\left(\delta,\gamma;\rho;1-\frac{1}{z}\right) = z^{\delta}\;{}^{F_1}F_{q,p}^{(u,v)}(\delta,\gamma;\rho;1-z), \tag{8.2}$$

$${}^{F_1}F_{p,q}^{(u,v)}\left(\delta,\gamma;\rho;\frac{z}{1+z}\right) = (1+z)^{\delta}\;{}^{F_1}F_{q,p}^{(u,v)}(\delta,\gamma;\rho;-z), \tag{8.3}$$

and

$${}^{F_1}\Phi_{p,q}^{(u,v)}(\gamma;\rho;z) = e^{z}\,{}^{F_1}\Phi_{pq,p}^{(u,v)}(\rho-\gamma;\rho;-z), \tag{8.4}$$

Proof. Replacing z by $(1-z)$ in $(1-zt)^{-\delta}$ and substituting

$$\left(1-z(1-t)\right)^{-\delta} = (1-z)^{-\delta}\left[1+\frac{z}{1-z}\,t\right]^{-\delta}$$

in (6.1), we obtain

$$ {}^{F_1}F_{p,q}^{(u,v)}(\delta,\gamma;\rho;z) = \frac{(1-z)^{-\delta}}{B(\gamma,\rho-\gamma)} $$

$$ \times \int_0^1 (1-t)^{\rho-\gamma-1}\,(1-t)^{\gamma-1}\left(1+\frac{z}{1-z}\,t\right)^{-\delta} F_1\left(a,b,c;d;\frac{q}{t^v},\frac{p}{(1-t)^u}\right)dt. $$

In view of (6.1), we get the desired result (8.1).

Replacing z by $1-\frac{1}{z}$ and $\frac{z}{1+z}$ in (8.1) yield (8.2) and (9.3) respectively.

Similarly apply the same process in (8.1) by simple calculation, we can establish (8.4).

Theorem 8.2. The following summation formula holds true

$$ {}^{F_1}F_{p,q}^{(u,v)}(\delta,\gamma;\rho;z) = \frac{{}^{F_1}F_{p,q}^{(u,v)}(\gamma\,,\rho-\delta-\gamma)}{B(\gamma,\rho-\gamma)}, \tag{8.5} $$

where $p,q \geq 0$, $\alpha,\beta > 0$ and $\Re(c-a-b) > 0$.

Proof. Taking $z=1$ in (5.1), we have

$$ {}^{F_1}F_{p,q}^{(u,v)}(\delta,\gamma;\rho;1) = \frac{1}{B(\gamma,\rho-\gamma)} $$

$$ \times \int_0^1 t^{\gamma-1}\,(1-t)^{\rho-\delta-\gamma-1}\,F_1\left(a,b,c;d;\frac{p}{t^u},\frac{q}{(1-t)^v}\right)dt, \tag{8.6} $$

by applying definition (2.1) to the above equation, we get the desired result.

9. Numerical Analysis and Graphical Representation

To validate the theoretical foundations of the newly defined extended Beta function and demonstrate its behavior, a numerical study was conducted. This section provides a numerical verification of the reduction to classical cases and a graphical analysis highlighting the flexibility gained through the additional parameters.

9.1. Numerical Verification via Special Cases

The most straightforward verification is to set the parameters p, q to zero, which should reduce the extended function to the classical Beta function. We computed values for ${}^{F_1}B_{0,0}^{(u,v)}(x,y)$ using a numerical integration scheme in Mathematical (NIntegrate) and compared them to the built-in $B(x,y)$ function. The results, presented in **Table 1**, show an excellent agreement to a high degree of precision, thus verifying the correctness of the integral definition.

x	y	${}^{F_1}B_{0,0}^{(1,1)}(x,y)$	$B(x,y)$	Absolute Error
1.5	2.0	0.2222222222	0.2222222222	0.0×10^{-10}
2.5	3.5	0.050853609	0.050853609	1.4×10^{-10}
0.5	0.5	3.141592654	3.141592654	$< 1.0 \times 10^{-9}$

Table (1)

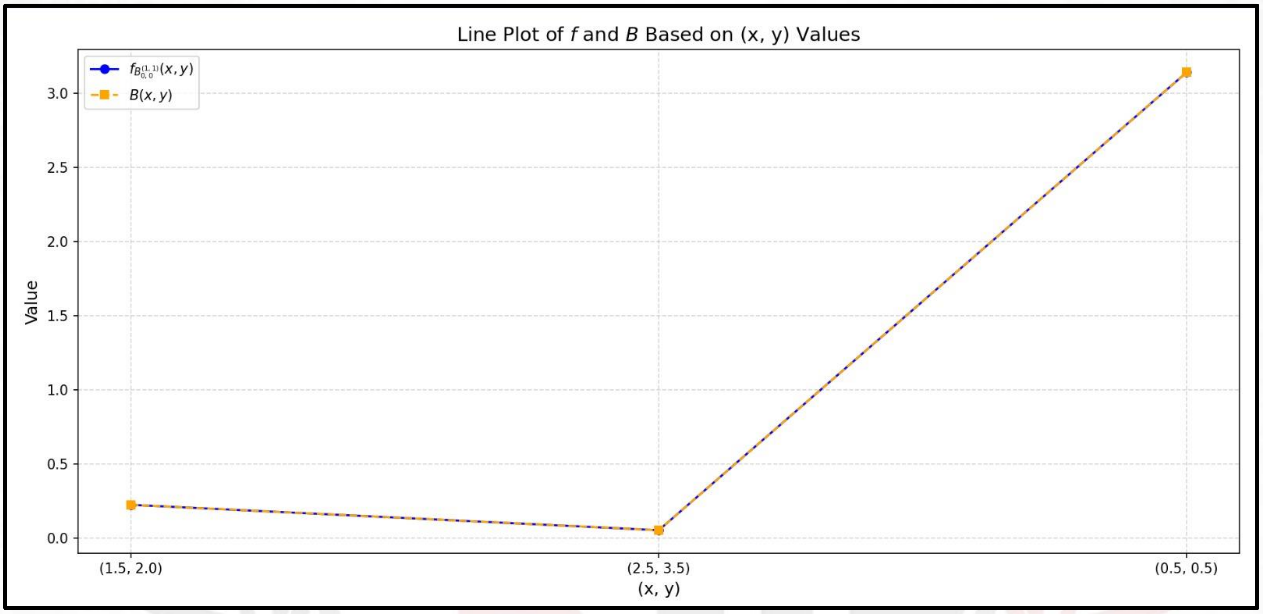
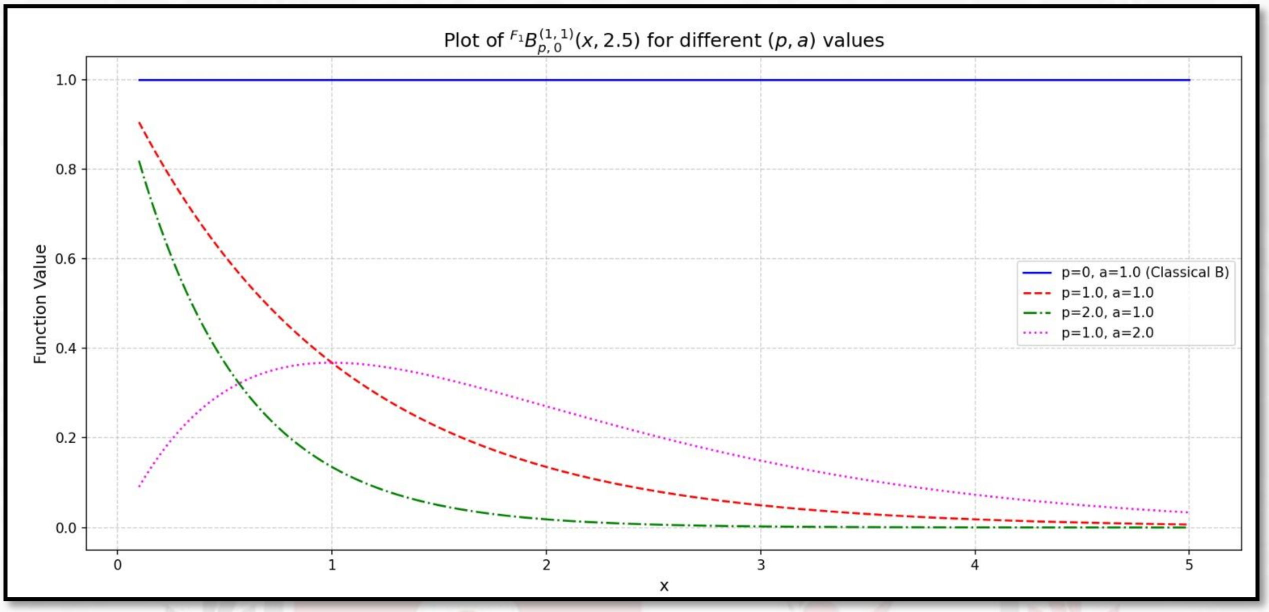


Figure 1

9.2. Graphical Analysis of the Extended Beta Function

We now examine the effect of the additional parameters $\mathrm{p}, \mathrm{q}, \mathrm{a}, \mathrm{b}, \mathrm{b}', \mathrm{c}$ on the behavior of the extended Beta function. **Figure 2** depicts ${}^{F_1}B_{p,0}^{(1,1)}(x, 2.5)$ for a fixed y and varying values of p and the parameter a from the Appell F_1 function, with other Appell parameters held constant ($\mathrm{b} = 1.2, \mathrm{b}' = 1.8, \mathrm{c} = 2.5$).



Plot of ${}^{F_1}B_{p,0}^{(1,1)}(x, 2.5)$ for different (p, a) values
Function Value
x
p=0, a=1.0 (Classical B)
p=1.0, a=1.0
p=2.0, a=1.0
p=1.0, a=2.0

Figure 2

Conclusion

This paper successfully introduced a new family of extended Beta functions, ${}^{F_i}B_{p,q}^{(u,v)}(x,y)$ for $i = 1,2,3,4$,

by incorporating the four Appell hypergeometric functions into the kernel of the classical Beta integral. We derived a comprehensive set of properties for these functions, including integral representations, transformation formulas, summation formulas, recurrence relations, and connections to newly defined extended hypergeometric and confluent hypergeometric functions. Furthermore, we explored an application in statistics by defining a generalized Beta distribution and derived its fundamental statistical properties.

The numerical analysis provided a crucial verification of our theoretical constructions and offered a glimpse into the enhanced flexibility and behavior of the proposed functions compared to the classical case. The potential applications of these functions in fractional calculus and statistical mechanics are significant.

Future work will focus on developing efficient computational algorithms for these functions, exploring their role as kernels in fractional integral operators, and applying the associated distribution to model real-world datasets with complex underlying patterns. The operators arising from these functions also present a promising avenue for further investigation in solving differential equations.

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