STARDOM UNIVERSITY

جامعة ستاردوم

مجلة ستاردوم العلمية للدراسات الطبيعية والهندسية

مجلة ستاردوم العلمية للدراسات الطبيعية والهندسية
تصدر بشكل نصف سنوي عن جامعة ستاردوم
المجلد الثاني ا العدد الأول- لعام 2024م
رقم الإيداع الدولي : ISSN 2980-3756

بسم الله الرحمن الرحيم

عناوين الأبحـــاث

Stardom Journal of Natural and Engineering Sciences (SJNES)

Introduction

Aims and Scope:

The stardom Journal for Natural and Engineering sciences is a journal for the publication of peer reviewed, original research for all aspects of Natural Sciences and Engineering Sciences.

This journal provides a forum for the discussion of Natural and Engineering sciences problems around the world and for the presentation of results. It is aimed only at the Natural and Engineering sciences.

Research Areas Include, but are not exclusive to:

- Biology and biomedical engineering
- Environmental Sciences
- Chemistry studies and Chemical Engineering
- Physics studies
- Material Studies
- Mathematics
- Computer Sciences and engineering
- Civil Engineering
- Area Engineering
- Environmental Engineering
- Telecommunications Engineering and networking
- Computer Engineering
- Electrical Engineering

- Agriculture Engineering
- Human-machine systems
- Space exploration
- Renewable energy
- Power systems and control
- Artificial intelligence and its applications
- Theoretical studies

Papers submitted should address Stardom Natural and Engineering sciences issues using a range of techniques e.g. case studies, observational and theoretical analyses, the application of science, engineering and technology to questions of environmental concern or mathematical and computer modeling techniques with the aim of informing both the researcher and practitioner.

Before you Begin

Conflict of interest

All authors are requested to disclose any actual or potential conflict of interest including any financial, personal or other relationships with other people or organizations within three years of beginning the submitted work that could inappropriately influence, or be perceived to influence, their work.

Submission declaration and verification

Submission of an article implies that the work described has not been published previously (except in the form of an abstract or as part of a published lecture or academic thesis or as an electronic preprint, that it is not under consideration for publication elsewhere, that its publication is approved by all authors and tacitly or explicitly by the responsible authorities where the work was carried out, and that, if accepted, it will not be published elsewhere in the same form, in English or in any other language, including electronically without the written consent of the copyright-holder.

Changes to authorship

This policy concerns the addition, deletion, or rearrangement of author names in the authorship of accepted manuscripts:

Before the accepted manuscript is published in an online issue: Requests to add or remove an author, or to rearrange the author names, must be sent to the Journal

Manager from the corresponding author of the accepted manuscript and must include: (a) the reason the name should be added or removed, or the author names rearranged and (b) written confirmation (e-mail, fax, letter) from all authors that they agree with the addition, removal or rearrangement. In the case of addition or removal of authors, this includes confirmation from the author being added or removed. Requests that are not sent by the corresponding author will be forwarded by the Journal Manager to the corresponding author, who must follow the procedure as described above. Note that: (1) Journal Managers will inform the Journal Editors of any such requests and (2) publication of the accepted manuscript in an online issue is suspended until authorship has been agreed.

After the accepted manuscript is published in an online issue: Any requests to add, delete, or rearrange author names in an article published in an online issue will follow the same policies as noted above and result in a corrigendum..

Role of the funding source

You are requested to identify who provided financial support for the conduct of the research and/or preparation of the article and to briefly describe the role of the sponsor(s), if any, in study design; in the collection, analysis and interpretation of data; in the writing of the report; and in the decision to submit the article for publication. If the funding source(s) had no such involvement then this should be stated.

Language

Please write your text in good English (American or British usage is accepted, but not a mixture of these). Authors who feel their English language manuscript may require editing to eliminate possible grammatical or spelling errors and to conform to correct scientific English may wish to use the English Language Editing

Submission

Submission to this journal proceeds totally online and you will be guided stepwise through the creation and uploading of your files. The system automatically converts source files to a single PDF file of the article, which is used in the peer-review process. Please note that even though manuscript source files are converted to PDF files at submission for the review process, these source files are needed for further processing after acceptance.

Referees

Please submit, with the manuscript, the names and addresses of at least three potential referees.

Preparation

Use of word-processing software

It is important that the file be saved in the native format of the word processor used. The text should be in single-column format. Keep the layout of the text as simple as possible. Most formatting codes will be removed and replaced on processing the article. In particular, do not use the word processor's options to justify text or to hyphenate words. However, do use bold face, italics, subscripts, superscripts etc. When preparing tables, if you are using a table grid, use only one grid for each individual table and not a grid for each row. If no grid is used, use tabs, not spaces, to align columns. The electronic text should be prepared in a way very similar to that of conventional manuscripts.

Note that source files of figures, tables and text graphics will be required whether or not you embed your figures in the text.

To avoid unnecessary errors, you are strongly advised to use the 'spell-check' and 'grammar-check' functions of your word processor.

Please **do not** add line numbering to the text document.

Article structure

Subdivision - numbered sections

Divide your article into clearly defined and numbered sections. Subsections should be numbered 1.1 (then 1.1.1, 1.1.2, ...), 1.2, etc. (the abstract is not included in section numbering). Use this numbering also for internal cross-referencing: do not just refer to 'the text'. Any subsection may be given a brief heading. Each heading should appear on its own separate line.

Introduction

State the objectives of the work and provide an adequate background, avoiding a detailed literature survey or a summary of the results.

Material and methods

Provide sufficient detail to allow the work to be reproduced. Methods already published should be indicated by a reference: only relevant modifications should be described.

Results

Results should be clear and concise.

Discussion

This should explore the significance of the results of the work, not repeat them. A combined Results and Discussion section is often appropriate. Avoid extensive citations and discussion of published literature.

Conclusions

The main conclusions of the study must be presented in a short Conclusions section, which may stand alone or form a subsection of a Discussion or Results and Discussion section.

Essential title page information

• **Title.** Concise and informative. Titles are often used in information-retrieval systems. Avoid abbreviations and formulae where possible.
• **Author names and affiliations.** Where the family name may be ambiguous (e.g., a double name), please indicate this clearly. Present the authors' affiliation addresses (where the actual work was done) below the names. Indicate all affiliations with a lower-case superscript letter immediately after the author's name and in front of the appropriate address. Provide the full postal address of each affiliation, including the country name and, if available, the e-mail address of each author.
• **Corresponding author.** Clearly indicate who will handle correspondence at all stages of refereeing and publication, also post-publication. **Ensure that phone numbers (with country and area code) are provided in addition to the e-mail address and the complete postal address. Contact details must be kept up to date by the corresponding author.**

• **Present/permanent address.** If an author has moved since the work described in the article was done, or was visiting at the time, a 'Present address' (or 'Permanent address') may be indicated as a footnote to that author's name. The address at which the author actually did the work must be retained as the main, affiliation address. Superscript Arabic numerals are used for such footnotes.

Abstract

A concise and factual abstract is required. The abstract should state briefly the purpose of the research, the principal results and major conclusions. An abstract is often presented separately from the article, so it must be able to stand alone. For this reason, References should be avoided, but if essential, then cite the author(s) and year(s). Also, non-standard or uncommon abbreviations should be avoided, but if essential they must be defined at their first mention in the abstract itself. Not more 250 words.

Graphical abstract

A Graphical abstract is optional and should summarize the contents of the article in a concise, pictorial form designed to capture the attention of a wide readership online. Authors must provide images that clearly represent the work described in the article. Graphical abstracts should be submitted as a separate file in the online submission system. Image size: Please provide an image with a minimum of 531 ×

1328 pixels (h × w) or proportionally more. The image should be readable at a size of 5 × 13 cm using a regular screen resolution of 96 dpi. Preferred file types: TIFF, EPS, PDF or MS Office files.

Highlights

High lights are mandatory for this journal. They consist of a short collection of bullet points that convey the core findings of the article and should be submitted in a separate file in the online submission system. Please use 'Highlights' in the file name and include 3 to 5 bullet points (maximum 85 characters, including spaces, per bullet point).

Keywords

Immediately after the abstract, provide a maximum of 6 keywords, using American spelling and avoiding general and plural terms and multiple concepts (avoid, for example, 'and', 'of'). Be sparing with abbreviations: only abbreviations firmly established in the field may be eligible. These keywords will be used for indexing purposes.

Acknowledgements

Collate acknowledgements in a separate section at the end of the article before the references and do not, therefore, include them on the title page, as a footnote to the title or otherwise. List here those individuals who provided help during the research (e.g., providing language help, writing assistance or proof reading the article, etc.).

Math formulae

Present simple formulae in the line of normal text where possible and use the solidus (/) instead of a horizontal line for small fractional terms, e.g., X/Y. In principle, variables are to be presented in italics. Powers of e are often more conveniently denoted by exp. Number consecutively any equations that have to be displayed separately from the text (if referred to explicitly in the text).

Footnotes

Footnotes should be used sparingly. Number them consecutively throughout the article, using superscript Arabic numbers. Many word processors build footnotes into the text, and this feature may be used. Should this not be the case, indicate the position of footnotes in the text and present the footnotes themselves separately at the end of the article. Do not include footnotes in the Reference list.

Table footnotes

Indicate each footnote in a table with a superscript lowercase letter.

Artwork

Electronic artwork

General points

• Make sure you use uniform lettering and sizing of your original artwork.
• Save text in illustrations as 'graphics' or enclose the font.
• Only use the following fonts in your illustrations: Arial, Courier, Times, Symbol.
• Number the illustrations according to their sequence in the text.
• Use a logical naming convention for your artwork files.
• Provide captions to illustrations separately.
• Produce images near to the desired size of the printed version.
• Submit each figure as a separate file.

Formats

Regardless of the application used, when your electronic artwork is finalized, please 'save as' or convert the images to one of the following formats (note the resolution requirements for line drawings, halftones, and line/halftone combinations given below):

EPS: Vector drawings. Embed the font or save the text as 'graphics'.
TIFF: Color or grayscale photographs (halftones): always use a minimum of 300 dpi.
TIFF: Bitmapped line drawings: use a minimum of 1000 dpi.
TIFF: Combinations bitmapped line/half-tone (color or grayscale): a minimum of 500 dpi is required.

If your electronic artwork is created in a Microsoft Office application (Word, PowerPoint, Excel) then please supply 'as is'.

Please do not:

• Supply files that are optimized for screen use (e.g., GIF, BMP, PICT, WPG); the resolution is too low;

• Supply files that are too low in resolution;

• Submit graphics that are disproportionately large for the content.

Color artwork

Please make sure that artwork files are in an acceptable format (TIFF, EPS or MS Office files) and with the correct resolution. If, together with your accepted. Please note: Because of technical complications which can arise by converting color figures to 'gray scale' (for the printed version should you not opt for color in print) please submit in addition usable black and white versions of all the color illustrations.

Figure captions

Ensure that each illustration has a caption. Supply captions separately, not attached to the figure. A caption should comprise a brief title (**not** on the figure itself) and a description of the illustration. Keep text in the illustrations themselves to a minimum but explain all symbols and abbreviations used.

Tables

Number tables consecutively in accordance with their appearance in the text. Place footnotes to tables below the table body and indicate them with superscript lowercase letters. Avoid vertical rules. Be sparing in the use of tables and ensure that the data presented in tables do not duplicate results described elsewhere in the article.

References

Citation in text

Please ensure that every reference cited in the text is also present in the reference list (and vice versa). Any references cited in the abstract must be given in full. Unpublished results and personal communications are not recommended in the reference list, but may be mentioned in the text. If these references are included in the reference list they should follow the standard reference style of the journal and should include a substitution of the publication date with either 'Unpublished results' or 'Personal communication'. Citation of a reference as 'in press' implies that the item has been accepted for publication.

Web references

As a minimum, the full URL should be given and the date when the reference was last accessed. Any further information, if known (DOI, author names, dates, reference to a source publication, etc.), should also be given. Web references can be listed separately (e.g., after the reference list) under a different heading if desired, or can be included in the reference list.

References in a special issue

Please ensure that the words 'this issue' are added to any references in the list (and any citations in the text) to other articles in the same Special Issue.

Reference style

Text: Indicate references by number(s) in square brackets in line with the text. The actual authors can be referred to, but the reference number(s) must always be given. Example: '..... as demonstrated [3,6]. Barnaby and Jones [8] obtained a different result ....'

List: Number the references (numbers in square brackets) in the list in the order in which they appear in the text.

Examples:
Reference to a journal publication:

[1] J. van der Geer, J.A.J. Hanraads, R.A. Lupton, The art of writing a scientific article, J. Sci. Commun. 163 (2010) 51–59.

Reference to a book:

[2] W. Strunk Jr., E.B. White, The Elements of Style, fourth ed., Longman, New York, 2000.

Reference to a chapter in an edited book:

[3] G.R. Mettam, L.B. Adams, How to prepare an electronic version of your article, in: B.S. Jones, R.Z. Smith (Eds.), Introduction to the Electronic Age, E-Publishing Inc., New York, 2009, pp. 281–304.

Submission checklist

The following list will be useful during the final checking of an article prior to sending it to the journal for review. Please consult this Guide for Authors for further details of any item.

Ensure that the following items are present:

One author has been designated as the corresponding author with contact details:
• E-mail address

• Full postal address
• Phone numbers
All necessary files have been uploaded, and contain:
• Keywords
• All figure captions
• All tables (including title, description, footnotes)
Further considerations
• Manuscript has been 'spell-checked' and 'grammar-checked'
• References are in the correct format for this journal
• All references mentioned in the Reference list are cited in the text, and vice versa
• Permission has been obtained for use of copyrighted material from other sources (including the Web)

• Color figures are clearly marked as being intended for color reproduction on the Web (free of charge) and in print, or to be reproduced in color on the Web (free of charge) and in black-and-white in print
• If only color on the Web is required, black-and-white versions of the figures are also supplied for printing purposes

A NEW EXTENDED BETA FUNCTION INVOLVING GENERALIZED MITTAG-LEFFLER FUNCTION AND IT'S APPLICATIONS

Dr. Salem Saleh Al-Qasemi Barahmah

Department of Mathematics, Faculty of Education-Aden, Aden University, Aden, Yemen

E-mail: salemalqasemi@yahoo.com

Abstract

The main object of this paper is to introduce a new extension of the beta function involving the generalized Mittag-leffler function and study its important properties, like integral representation, summation formula, derivative formula, beta distribution and transform formula. We introduce new extended hypergeometric and confluent hypergeometric functions.

Keywords: Beta function, Beta Distribution, Confluent hypergeometric function, Gamma function, Hypergeometric function, Summation formulas, Transform formula

Introduction:

There are many extensions and generalizations of the beta function, hypergeometric function and confluent hypergeometric function have been considered by several authors (see [1, 2, 4, 5, 6, 7, 8, 9, 13, 14, 15]). In this paper, we study another extension of the Euler Beta function and investigate various formulas, such as integral representation, summation formula and derivative formula. Further, we obtain beta distribution and its some statistical formulas. We extend also the definition of hypergeometric and confluent hypergeometric functions and study their various properties.

The classical Gauss hypergeometric function (see [17]) is defined as

$$F(\delta_1, \delta_2; \delta_3; \tau) = \sum_{n=0}^{\infty} \frac{(\delta_1)_n (\delta_2)_n}{(\delta_3)_n} \frac{\tau^n}{n!}, \tag{1.1}$$

where $(\delta)_n$ $(\delta \in \mathbb{C})$ is the Pochhammer symbol defined by

$$(\delta)_n = \frac{\Gamma(\delta + n)}{\Gamma(\delta)} . \tag{1.2}$$

The confluent hypergeometric function (see [17]) is defined by

$$\Phi(\delta_1; \delta_2; \tau) = \sum_{n=0}^{\infty} \frac{(\delta_1)_n}{(\delta_2)_n} \frac{\tau^n}{n!} . \tag{1.3}$$

The Gamma function $\Gamma(\tau)$ developed by Euler [3] with the intent to extend the factorials to values between the integers is defined by the definite integral

$$\Gamma(z) = \int_0^{\infty} e^{-t} \, t^{z-1} \, dt \quad , \quad Re(z) > 0 . \tag{1.4}$$

Among various extensions of gamma function, we mention here the extended gamma function [4] defined by Chaudhry and Zubair

$$\Gamma_p(z) = \int_0^\infty t^{z-1} \exp\left(-t - \frac{p}{t}\right) dt \quad , \quad (Re(p) > 0). \tag{1.5}$$

The Euler Beta function $B(z_1, z_2)$ (see [3]) is defined by

$$B(z_1, z_2) = \int_0^1 t^{z_1-1} (1-t)^{z_2-1} \; dt \tag{1.6}$$

$$= \frac{\Gamma(z_1)\Gamma(z_2)}{\Gamma(z_1+z_2)} = \frac{(z_1-1)! \; (z_2-1)!}{(z_1+z_2-1)!}, \tag{1.7}$$

where $z! = \Gamma(z+1)$, $z = 0,1,2,4, \dots$, $(Re(z_1) > 0 \; , Re(z_2) > 0)$.

In 1997, Choudhary et al. [5] introduced an extension of the beta function defined by

$$B^p(z_1, z_2) = \int_0^1 t^{z_1-1} (1-t)^{z_2-1} \, exp\left(-\frac{p}{t(1-t)}\right) dt, \tag{1.8}$$

where $Re(p) \geq 0$, $(Re(z_1) > 0 \; , Re(z_2) > 0)$.

Chaudhary et al. [6], used the new extended the beta function $B^p(\delta_1, \delta_2)$ to introduce an extended hypergeometric and confluent hypergeometric functions defined respectively as

$$F^p(\delta_1, \delta_2, \delta_3; \tau) = \sum_{n=0}^{\infty} (\delta_1)_n \frac{B^p(\delta_1+n, \delta_3-\delta_2)}{B(\delta_2, \delta_3-\delta_2)} \frac{\tau^n}{n!}, \tag{1.9}$$

$$(p \geq 0 , \; |\tau| < 1 , Re(\delta_1) > 0 , \; Re(\delta_3) > Re(\delta_2) > 0),$$

and

$$\Phi^p(\delta_2; \delta_3; \tau) = \sum_{n=0}^{\infty} \frac{B^p(\delta_2+n, \delta_3-\delta_2)}{B(\delta_2, \delta_3-\delta_2)} \frac{\tau^n}{n!}, \tag{1.10}$$

$$(p \geq 0 , |\tau| < 1 , \; Re(\delta_3) > Re(\delta_2) > 0).$$

In 2018, Shadab et al. [15] introduced an extended the beta function in terms of the classical Mittag-Leffler function defined as

$$B_\alpha^p(\delta_1, \delta_2) = \int_0^1 t^{\delta_1-1} (1-t)^{\delta_2-1} E_\alpha exp\left(-\frac{p}{t(1-t)}\right) dt, \tag{1.11}$$

$$Re(p) \geq 0 , Re(\delta_1) > 0 , Re(\delta_2) > 0, \qquad \alpha \in \mathbb{R}_0^+,$$

where $E_\alpha(\cdot)$ is the classical Mittag-Leffler function defined as [10]

$$E_\alpha(x) = \sum_{n=0}^{\infty} \frac{\tau^n}{\Gamma(\alpha n+1)}, \qquad x \in C \ , \alpha \in \mathbb{R}_0^+$$

Shadab et al. [15], used the extended Beta function $B_\alpha^p(\delta_1, \delta_2)$ to introduce a new extended hypergeometric and confluent hypergeometric functions defined respectively as

$$F_{p,\alpha}(\delta_1, \delta_2, \delta_3; \tau) = \sum_{n=0}^{\infty} (\delta_1)_n \frac{B_\alpha^p(\delta_1+n, \delta_3-\delta_2)}{B(\delta_2, \delta_3-\delta_2)} \frac{\tau^n}{n!}. \tag{1.12}$$

$$(p \in \mathbb{R}_0^+, \ , \ \alpha \in \mathbb{R}^+ , \ |\tau| < 1, \ Re(\delta_1) > 0 , \ Re(\delta_3) > Re(\delta_2) > 0).$$

The confluent hypergeometric function is defined as Φ

$$\Phi_{p,\alpha}(\delta_2; \delta_3; \tau) = \sum_{n=0}^{\infty} \frac{B_\alpha^p(\delta_2+n, \delta_3-\delta_2)}{B(\delta_2, \delta_3-\delta_2)} \frac{\tau^n}{n!}, \tag{1.13}$$

$$(p \in \mathbb{R}_0^+, \ , \ \alpha \in \mathbb{R}^+ , \ |\tau| < 1, \ Re(\delta_3) > Re(\delta_2) > 0).$$

In 2018, Al-Gonah et al. [1], introduced a new extended Beta function in terms of the classical Mittag-Leffler function defined as

$$B_{p,\alpha}^\lambda(a,b) = \int_0^1 t^{a-1} (1-t)^{b-1} E_{p,\alpha}^\lambda\left(-\frac{p}{t(1-t)}\right) dt, \tag{1.14}$$

$$Re(p) \geq 0 , Re(\alpha) > 0 \ \ Re(\lambda) > 0, Re(a) > 0, Re(b) > 0$$

Al-Gonah et al. [2] used the extended Beta function $B_{p,\alpha}^\lambda(a,b)$ to introduce a new extended hypergeometric and confluent hypergeometric functions defined respectively as

$$F_{p,\alpha}^\lambda(a,b;c;z) = \sum_{n=0}^{\infty} (\delta_1)_n \frac{B_{p,\alpha}^\lambda(a+n, c-b)}{B(b, c-b)} \frac{z^n}{n!}, \tag{1.15}$$

$$Re(p) \geq 0 , |z| < 1 , Re(\alpha) > 0, Re(\lambda) > 0, Re(a) > 0, Re(b) > \ Re(c) > 0.$$

$$\Phi_{p,\alpha}^\lambda(b;c;z) = \sum_{n=0}^{\infty} \frac{\mathrm{B}_{p,\alpha}^\lambda(c+n, c-b)}{B(b, c-b)} \frac{z^n}{n!}, \tag{1.16}$$

$$Re(p) \geq 0 , |z| < 1 , Re(\alpha) > 0, Re(\lambda) > 0, Re(b) > \ Re(c) > 0.$$

$$E_{\alpha,\beta}^\gamma \ (x) = \sum_{k=0}^{\infty} \frac{(\gamma)_k}{\Gamma(\alpha k+\beta)} \frac{x^k}{k!}, \alpha, \beta, \gamma \ \epsilon \mathbb{C}, \qquad Re\ (\alpha) , Re\ (\beta) , Re\ (\gamma) \ > 0 \, .$$

In 2020, Oraby and Rizq [12], introduced a new extended Beta function in terms of the classical Mittag-Leffler function defined as

$$B_{p,\alpha}^{\lambda,m}(a,b) = \int_0^1 t^{a-1}\,(1-t)^{b-1} E_{\lambda,\alpha}\left(-\frac{p}{t^m(1-t)^m}\right)dt, \qquad (1.17)$$

$$Re(p) \geq 0\,, Re(\alpha) > 0\;\; Re(\lambda) > 0, Re(m) > 0\,, Re(a) > 0, Re(b) > 0$$

Oraby et al. [12] used the extended Beta function $B_{p,\alpha}^{\lambda,m}(a,b)$ to introduce a new extended hypergeometric and confluent hypergeometric functions defined respectively as

$$F_{p,\alpha}^{\lambda,m}(a,b;c;z) = \sum_{n=0}^{\infty}(\delta_1)_n\,\frac{B_{p,\alpha}^{\lambda,m}(a+n,c-b)}{B(b,c-b)}\,\frac{z^n}{n!}, \qquad (1.18)$$

$$\left(\begin{matrix} Re(p) \geq 0\,, |z| < 1\,, Re(\alpha) > 0\;\; Re(\lambda) > 0, Re(m) > 0\,,\;\; Re(a) > 0, \\ Re(b) > \;\; Re(c) > 0 \end{matrix}\right).$$

$$\Phi_{p,\alpha}^{\lambda,m}(b;c;z) = \sum_{n=0}^{\infty}\frac{\mathrm{B}_{p,\alpha}^{\lambda,m}(c+n,c-b)}{B(b,c-b)}\,\frac{z^n}{n!}, \qquad (1.19)$$

$$Re(p) \geq 0\,, |z| < 1\,, Re(\alpha) > 0\;\; Re(\lambda) > 0, Re(m) > 0\,, Re(b) > 0, Re(b) > 0.$$

In 2022, Khan et al. [9] introduced a new extended Beta function in terms of the classical Mittag-Leffler function defined as

$$B_{\alpha,\beta}^{p,\mu,v}(\theta_1,\theta_2) = \int_0^1 t^{\theta_1-1}\,(1-t)^{\theta_2-1} E_{\alpha,\beta}\left(-\frac{p}{t^u(1-t)^v}\right)dt, \qquad (1.20)$$

$$Re(p) \geq 0, \qquad Re(\theta_1) > 0\,, Re(\theta_2) > 0, \qquad \alpha,\beta \in \mathbb{R}_0^+\,, \mu, v \in \mathbb{R}^+.$$

where $E_{\alpha,\beta}(.)$ is the generalized Mittag-Leffler function defined as [11]

$$E_{\alpha,\beta}(x) = \sum_{n=0}^{\infty}\frac{x^n}{\Gamma(\alpha n+\beta)}\,, \quad x \in \mathbb{C}\;, \alpha,\beta \in \mathbb{R}_0^+\,.$$

Khan et al. [9] used the extended Beta function $B_{\alpha,\beta}^{p,\mu,v}(\theta_1,\theta_2)$ to introduce a new

extended hypergeometric and confluent hypergeometric functions defined respectively as

$$F_{p,\alpha}^{\lambda,m}(\theta_1,\theta_2;\theta_2;\omega)=\sum_{n=0}^{\infty}(\theta_1)_n\ \frac{B_{p,\alpha}^{\lambda,m}(\theta_2+n,\theta_3-\theta_2)}{B(\theta_2,\theta_3-\theta_2)}\ \frac{\omega^n}{n!}, \tag{1.21}$$

$Re(p)\geq 0\,,|\omega|<1\,,Re(\alpha)>0\ \ Re(\lambda)>0,Re(m)>0\,,$

$Re(\theta_1)>0,Re(\theta_2)>0,Re(\theta_3)>0$

$$\Phi_{p,\alpha}^{\lambda,m}(\theta_2;\theta_2;\omega)=\sum_{n=0}^{\infty}\frac{\mathrm{B}_{p,\alpha}^{\lambda,m}(\theta_2+n,\theta_3-\theta_2)}{B(\theta_2,\theta_3-\theta_2)}\ \frac{\omega^n}{n!}, \tag{1.22}$$

$p\geq 0\,,|\omega|<1\,,Re(\alpha)>0\ \ Re(\lambda)>0,Re(m)>0\,,Re(\theta_2)>0,Re(\theta_3)>0.$

2. A new extension of the beta function

In this section, we introduce a new extension of the extended Beta function and investigate various properties and representations

$$B_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(\delta_1,\delta_2)=\int_0^1 t^{\delta_1-1}\,(1-t)^{\delta_2-1}E_{\alpha,\beta}^{\gamma,\sigma}\left(-\frac{p}{t^u(1-t)^v}\right)dt, \tag{2.1}$$

$Re(p)>0,Re(\delta_1)>0\,,Re(\delta_2)>0,\qquad \alpha,\beta,\gamma,\sigma\in\mathbb{R}_0^+\,,\mu,v\in\mathbb{R}^+,$

where $E_{\alpha,\beta}^{\gamma,\sigma}(.)$ is the generalized Mittag-Leffler function defined in [15].

If $\sigma=1$ in (2.1), we get the new result

$$B_{\alpha,\beta}^{(p,\mu,v,\gamma,1)}(\delta_1,\delta_2)=\int_0^1 t^{\delta_1-1}\,(1-t)^{\delta_2-1}E_{\alpha,\beta}^{\gamma}\left(-\frac{p}{t^u(1-t)^v}\right)dt. \tag{2.2}$$

If $\sigma=\gamma=1$, in (2.1), then (2.1) reduce to (1.20)

$$B_{\alpha,\beta}^{(p,\mu,v,1,1)}(\delta_1,\delta_2)=\int_0^1 t^{\delta_1-1}\,(1-t)^{\delta_2-1}E_{\alpha,\beta}^{\square}\left(-\frac{p}{t^u(1-t)^v}\right)dt. \tag{2.3}$$

If $\sigma=\gamma=1$ and $\mu=v=m$ then (2.1) reduces to extended beta function (1.17)

$$B_{\alpha,\beta}^{(p,m)}(\delta_1,\delta_2)=\int_0^1 t^{\delta_1-1}\,(1-t)^{\delta_2-1}E_{\alpha,\beta}^{\square}\left(-\frac{p}{t^m(1-t)^m}\right)dt. \tag{2.4}$$

If $\sigma=1$ and $u=v=1$, then (2.1) reduces to extended beta function (1.14)

$$B_{\alpha,\beta}^{(p,\gamma)}(\delta_1,\delta_2)=\int_0^1 t^{\delta_1-1}\,(1-t)^{\delta_2-1}E_{\alpha,\beta}^{\gamma}\left(-\frac{p}{t(1-t)}\right)dt. \qquad (2.5)$$

If $\gamma=\sigma=\beta=u=v=1$, then (2.1) reduces to extended beta function (1.11).

$$B_{\alpha,1}^{(p,1,1,1,1)}(\delta_1,\delta_2)=B_{\alpha}^{p}(\delta_1,\delta_2)=B_{\alpha}(\delta_1,\delta_2;p). \qquad (2.6)$$

If $\gamma=\sigma=\alpha=\beta=u=v=1$, then (2.1) reduces to extended beta function (1.8).

$$B_{1,1}^{(p,1,1,1,1)}(\delta_1,\delta_2)=B^{p}(\delta_1,\delta_2)=B(\delta_1,\delta_2;p). \qquad (2.7)$$

3. Properties of $B_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(\delta_1,\delta_2)$

In this section, we present certain properties of extension Beta function including summation formulas, integral representations and Mellin transform.

Theorem 3.1. The following integral representations hold

$$B_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(\delta_1,\delta_2)=2\int_0^{\frac{\pi}{2}}\cos^{2\delta_1-1}\theta\,\sin^{2\delta_2-1}\theta\,E_{\alpha,\beta}^{\gamma,\sigma}(-p(\sec^2\theta)^{\mu}(cosec^2\theta)^{v})d\theta, \qquad (3.1)$$

$$B_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(\delta_1,\delta_2)=\int_0^{\infty}\frac{u^{\delta_1-1}}{(1+u)^{\delta_1+\delta_2}}\,E_{\alpha,\beta}^{\gamma,\sigma}\left(-p\,\frac{(1+u)^{\mu+v}}{u^{\mu}}\right)du, \qquad (3.2)$$

$$B_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(\delta_1,\delta_2)=2^{1-\delta_1-y}\int_{-1}^{1}(1-u)^{\delta_1-1}(1-u)^{\delta_2-1}\times E_{\alpha,\beta}^{\gamma,\sigma}\left(-p\,\frac{2^{\mu+v}}{(1-u)^{\mu}(1-u)^{v}}\right)du\,, \qquad (3.3)$$

$$Re(p)>0, Re(\delta_1)>0\,, Re(\delta_2)>0, \qquad \alpha,\beta,\gamma,\sigma\in\mathbb{R}^+\,,\mu,v\in\mathbb{R}^+.$$

Proof. Let $t=\cos^2\theta$, $t=\frac{u}{1+u}$, $t=\frac{1+u}{2}$, respectively in equation (2.1), we obtain the above representations.

Remark 3.1. If we take $\gamma=\sigma=1$, in the integral representation of the Theorem (3.1), we obtain corresponding integrals for $B_{\alpha,\beta}^{(p,\mu,v)}(\delta_1,\delta_2)$ in [9].

If we take $\gamma = \sigma = 1 \ and \ \mu = v = m$, in the integral representation of the Theorem (3.1), we obtain corresponding integrals for $B_{\alpha,\beta}^{(p,m)}(\delta_1, \delta_2)$ in [12].

If we take $\sigma = 1 \ and \ \mu = v = 1$, in the integral representation of the Theorem (3.1), we obtain corresponding integrals for $B_{\alpha,\beta}^{\gamma}(\delta_1, \delta_2)$ in [1].

If we take $\gamma = 1, \sigma = 1, \beta = 1, \mu = 1, v = 1$, in the integral representation of the Theorem (3.1), we obtain corresponding integrals for $B_{\alpha}(\delta_1, \delta_2; p)$ in [15].

If we take $\gamma = 1, \sigma = 1, \alpha = 1, \beta = 1, \mu = 1, v = 1$, in the integral representation of the Theorem (3.1), we obtain corresponding integrals for $B(\delta_1, \delta_2; p)$ in [5].

Theorem 3.2. The following summation formula for $B_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(\delta_1, \delta_2)$ holds

$$B_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(\delta_1, \delta_2) = \sum_{k=0}^{n} \binom{n}{k} B_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(\delta_1 + k, \delta_2 + n - k), n \in N_0. \quad (3.4)$$

Proof. We find from (2.1) that

$$B_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(\delta_1, \delta_2) = \int_0^1 t^{\delta_1 - 1} (1-t)^{\delta_2 - 1} [t + (1-t)] E_{\alpha,\beta}^{\gamma,\sigma}\left(-\frac{p}{t^u (1-t)^v}\right) dt$$
$$= B_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(\delta_1 + 1, \delta_2) + B_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(\delta_1, \delta_2 + 1). \quad (3.5)$$

Repeating the same argument to the above two terms in (3.5), we obtain

$$B_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(\delta_1, \delta_2) = B_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(\delta_1 + 2, \delta_2)$$
$$+ 2B_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(\delta_1 + 1, \delta_2 + 1) + B_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(\delta_1, \delta_2 + 1). \quad (3.6)$$

Continuing this process, by using mathematical induction we get the desired result (3.4).

Theorem 3.3. The following summation formula for $B_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(\delta_1, \delta_2)$ holds

$$B_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(\delta_1, \delta_2) = \sum_{k=0}^{\infty} \frac{(\delta_2)_n}{n!} B_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(\delta_1 + n, 1) \quad n \in N_0 \quad (3.7)$$
$$Re(p) > 0, Re(\delta_1) > 0, Re(\delta_2) > 0, \quad \alpha, \beta, \gamma, \sigma \in \mathbb{R}^+, \mu, v \in \mathbb{R}^+.$$

Proof. To prove the above result, by using the generalized binomial theorem defined as

$$(1-t)^{-y} = \sum_{n=0}^{\infty} (y)_n \frac{t^n}{n!} \qquad (|t| < 1). \tag{3.8}$$

We fined

$$B_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(\delta_1, \delta_2) = \int_0^1 \sum_{n=0}^{\infty} (\delta_2)_n \frac{t^{\delta_1 n-1}}{n!} E_{\alpha,\beta}^{\gamma,\sigma}\left(-\frac{p}{t^u(1-t)^v}\right) dt. \tag{3.9}$$

Interchanging the order of integral and summation in the above equation and using (2.1), we get the desired result (3.7).

Theorem 3.4. The following summation formula for $B_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(\delta_1, \delta_2)$ holds

$$B_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(\delta_1, \delta_2) = \sum_{k=0}^{\infty} B_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(\delta_1 + 1, \delta_2 + 1) \tag{3.10}$$

$$Re(p) > 0\,, Re(\delta_1) > 0\,, Re(\delta_2) > 0, \qquad \alpha, \beta, \gamma, \sigma \in \mathbb{R}^+\,, \mu, v \in \mathbb{R}^+.$$

Proof. Using the relation

$$(1-t)^{y-1} = (1-t)^y \sum_{n=0}^{\infty} t^n \qquad (|t| < 1) \tag{3.11}$$

in (2.1), we obtain

$$B_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(\delta_1, \delta_2) = \int_0^1 (1-t)^{\delta_2} \sum_{n=0}^{\infty} t^{n+\delta_1-1} E_{\alpha,\beta}^{\gamma,\sigma}\left(-\frac{p}{t^u(1-t)^v}\right) dt,$$

$$B_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(\delta_1, \delta_2) = \sum_{n=0}^{\infty} \int_0^1 (1-t)^{\delta_2} t^{n+\delta_1-1} E_{\alpha,\beta}^{\gamma,\sigma}\left(-\frac{p}{t^u(1-t)^v}\right) dt,$$

which in view of (2.1), we get the desired result (3.10).

Remark 3.2.

In case $\gamma = 1\,, \sigma = 1,$ of (3.4) for $n = 1$, then (3.7) and (3.10) reduces to corresponding results in [9].

In case $\gamma = 1\,, \sigma = 1$ *and* $\mu = v = m$ of (3.4) for $n = 1$, then (3.7) and (3.10) reduces to corresponding results in [12].

In case $\sigma = 1$ *and* $\mu = v = 1$ of (3.4) for $n = 1$, then (3.7) and (3.10) reduces to corresponding results in [1].

In case $\gamma = 1, \sigma = 1, \beta = 1, \mu = 1, v = 1$ of (3.4) for $n = 1$, then (3.7) and (3.10) reduces to corresponding results in [15].

In case $\gamma = 1, \sigma = 1, \alpha = 1, \beta = 1, \mu = 1, v = 1$ of (3.4) for $n = 1$, then (3.7) and (3.10) reduces to corresponding results in [5].

In case $\sigma = 1$ of (3.4) for $n = 1$, (3.7) and (3.10) ,we get the following new results

$$B_{\alpha,\beta}^{(p,\mu,v,\gamma)}(\delta_1, \delta_2) = \sum_{k=0}^{\infty} \frac{(\delta_2)_n}{n!} B_{\alpha,\beta}^{(p,\mu,v,\gamma)}(\delta_1 + n, 1) \quad n \in N_0, \tag{3.12}$$

and

$$B_{\alpha,\beta}^{(p,\mu,v,\gamma)}(\delta_1, \delta_2) = \sum_{k=0}^{n} B_{\alpha,\beta}^{(p,\mu,v,\gamma)}(\delta_1 + 1, \delta_2 + 1). \tag{3.13}$$

4. Beta distribution of $B_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(\delta_1, \delta_2)$

We now define the beta distribution of (2.1), and obtain its mean, variance, moment generating function and cumulative distribution.

For $B_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(\delta_1, \delta_2)$, the Beta distribution is given by

$$f(t) = \begin{cases} \frac{1}{B_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(\delta_1,\delta_2)} t^{\delta_1 - 1} (1-t)^{\delta_2 - 1} E_{\alpha,\beta}^{\gamma,\sigma}\left(-\frac{p}{t^u (1-t)^v}\right) & (0 < t < 1), \\ 0, & otherwise \end{cases} \tag{4.1}$$

$$\delta_1, \delta_2 \epsilon \mathbb{R}, \quad \alpha, \beta, \gamma, \sigma \in \mathbb{R}^+, \quad \mu, v \in \mathbb{R}^+.$$

For $d \in R$, the d^{th} moment of a random variable X defined as

$$\rho = E(X^d) = \frac{B_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(\delta_1 + d, \delta_2)}{B_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(\delta_1, \delta_2)}, \tag{4.2}$$

$$\delta_1, \delta_2 \in \mathbb{R}, \quad p \geq 0, \quad \alpha, \beta, \gamma, \sigma \in \mathbb{R}^+, \quad \mu, v \in \mathbb{R}^+.$$

The variance of the distribution is defined by

$$\sigma^2 = E(X^2) - \left(E(X)\right)^2$$

$$= \frac{B_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(\delta_1, \delta_2) + B_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(\delta_1 + 2, \delta_2) - \left\{B_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(\delta_1 + 1, \delta_2)\right\}^2}{\left\{B_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(\delta_1, \delta_2)\right\}^2}. \tag{4.3}$$

The moment generating function of the distribution is defined as

$$M(t)=\sum_{n=0}^{\infty}\frac{t^n}{n!}\,E(X^n)=\frac{1}{B_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(\delta_1,\delta_2)}\sum_{n=0}^{\infty}B_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(\delta_1+n,\delta_2)\,\frac{t^n}{n!}. \quad (4.4)$$

The cumulative distribution is defined as

$$f(z)=\frac{B_{z,\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(\delta_1+d,\ \delta_2)}{B_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(\delta_1,\delta_2)}. \quad (4.5)$$

where

$$B_{z,\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(\delta_1,\delta_2)=\int_0^z t^{\delta_1-1}\,(1-t)^{\delta_2-1}E_{\alpha,\beta}^{\gamma,\sigma}\left(-\frac{p}{t^u(1-t)^v}\right)dt, \quad (4.6)$$

$$(p>0,\quad -\infty<\mu,v<\infty),$$

is the extended incomplete Beta function.

5-Generalization of extended hypergeometric and confluent hypergeometric functions

Here, we introduce a generalization of extended hypergeometric and confluent hypergeometric functions in terms of $B_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(\delta_1,\delta_2)$.

The extended hypergeometric function is defined as

$$F_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(\delta_1,\delta_2,\delta_3;\tau)=\sum_{n=0}^{\infty}(\delta_1)_n\,\frac{B_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(\delta_1+n,\delta_3-\delta_2)}{B(\delta_2,\delta_3-\delta_2)}\,\frac{\tau^n}{n!}, \quad (5.1)$$

$$(p\geq 0\,,\ \ |\tau|<1\,,\ \ \alpha,\beta,\gamma,\sigma,\mu,v>0\,,\ \ Re(\delta_3)>Re(\delta_2)>0).$$

The confluent hypergeometric function is defined as

$$\Phi_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(\delta_2;\delta_3;\tau)=\sum_{n=0}^{\infty}\frac{B_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(\delta_2+n,\delta_3-\delta_2)}{B(\delta_2,\delta_3-\delta_2)}\,\frac{\tau^n}{n!}, \quad (5.2)$$

$$(p\geq 0\,,\ \ \alpha,\beta,\gamma,\sigma,\mu,v>0\,,\ \ Re(\delta_3)>Re(\delta_2)>0).$$

Remark 5.1. In case $\alpha=\beta=\sigma=\gamma=\mu=v=1$ in (5.1) and (5.2), we obtain corresponding results in [6].

In case $\sigma=\gamma=1$ in (5.1) and (5.2), we obtain corresponding result in [9].

In case $\sigma=\gamma=1$ and $\mu=v=1$ in (5.1) and (5.2), we obtain corresponding result in [12].

In case $\sigma = \mu = v = 1$ in (5.1) and (5.2), we obtain corresponding result in [2].
In case $\sigma = \beta = \mu = v = 1$ in (5.1) and (5.2), we obtain corresponding result in [15].
In case $\sigma = 1$ in (5.1) and (5.2), we get the following new results

$$F_{\alpha,\beta}^{(p,\mu,v,\gamma)}(\delta_1,\delta_2,\delta_3;\tau) = \sum_{n=0}^{\infty}(\delta_1)_n \frac{B_{\alpha,\beta}^{(p,\mu,v,\gamma)}(\delta_1+n,\delta_3-\delta_2)}{B(\delta_2,\delta_3-\delta_2)} \frac{\tau^n}{n!}, \tag{5.3}$$

$$(p \geq 0\,,\ |\tau| < 1\,,\ \alpha,\beta,\gamma,\mu,v > 0\,,\ Re(\delta_3) > Re(\delta_2) > 0)$$

and

$$\Phi_{\alpha,\beta}^{(p,\mu,v,\gamma)}(\delta_2;\delta_3;\tau) = \sum_{n=0}^{\infty} \frac{B_{\alpha,\beta}^{(p,\mu,v,\gamma)}(\delta_2+n,\delta_3-\delta_2)}{B(\delta_2,\delta_3-\delta_2)} \frac{\tau^n}{n!}, \tag{5.4}$$

$$(p \geq 0\,,\ \alpha,\beta,\gamma,\mu,v > 0\,,\ Re(\delta_3) > Re(\delta_2) > 0).$$

6. Integral Representation and derivative formula for extended Gauss hypergeometric functions

Theorem 6.1. The following integral representations for the extended hypergeometric function $F_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(\delta_1,\delta_2,\delta_3;\tau)$ and confluent hypergeometric function $\Phi_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(\delta_2;\delta_3;\tau)$ holds

$$F_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(\delta_1,\delta_2,\delta_3;\tau) = \frac{1}{B(\delta_2,\delta_3-\delta_2)}$$

$$\int_{-1}^{1} t^{\delta_2-1}(1-t)^{\delta_3-\delta_2-1}(1-\tau t)^{-\delta_1} E_{\alpha,\beta}^{\gamma,\sigma}\left(-p\ \frac{2^{\mu+v}}{(1-u)^{\mu}(1-u)^{v}}\right), \tag{6.1}$$

$$(p \in \mathbb{R}_0^+,\ \alpha,\beta,\gamma,\sigma,\mu,v \in \mathbb{R}^+\ ;\ and\ \ arg|1-\tau| < \pi,\ Re(\delta_3) > Re(\delta_2) > 0).$$

$$\Phi_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(\delta_2;\delta_3;\tau) = \frac{1}{B(\delta_2,\delta_3-\delta_2)}$$

$$\times \int_{-1}^{1} t^{\delta_2-1}(1-t)^{\delta_3-\delta_2-1} e^{zt}\, E_{\alpha,\beta}^{\gamma,\sigma}\left(-p\ \frac{2^{\mu+v}}{(1-u)^{\mu}(1-u)^{v}}\right), \tag{6.2}$$

$$(p \in \mathbb{R}_0^+,\ \alpha,\beta,\gamma,\sigma,\mu,v \in \mathbb{R}^+\ ;\quad Re(\delta_3) > Re(\delta_2) > 0).$$

Proof. By using the definition of $B_{z,\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(x,y)$ in (2.1) into (5.1) and interchanging the order of integration and summation, which is verified under the condition here, we have

$$F_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(\delta_1,\delta_2,\delta_3;\tau) = \frac{1}{B(\delta_2,\delta_3-\delta_2)} \int_{-1}^{1} t^{\delta_2-1}(1-t)^{\delta_3-\delta_2-1}$$
$$\times E_{\alpha,\beta}^{\gamma,\sigma}\left(-p\ \frac{2^{\mu+v}}{(1-u)^{\mu}(1-u)^{v}}\right) \sum_{n=0}^{\infty} (\delta_1)_n \frac{(\tau t)^n}{n!}. \qquad (6.3)$$

Using the binomial theorem in (3.11) to the summation formula in (6.3), we get the desired result (6.1). Similarly, we can obtain (6.2).

Remark 6.1. In case $\alpha = \beta = \sigma = \gamma = \mu = v = 1$ in (6.1) and (6.2), we obtain the corresponding result in [6].
In case $\beta = \sigma = \gamma = \mu = v = 1$ in (6.1) and (6.2), we obtain the corresponding result in [15].
In case $\sigma = \mu = v = 1$ in (6.1) and (6.2), we obtain the corresponding result in [2].
In case $\sigma = \gamma = 1$ and $\mu = v = m$ in (6.1) and (6.2), we obtain the corresponding result in [12].
In case $\sigma = \gamma = 1$ in (6.1) and (6.2), we obtain the corresponding result in [9].
In case $\sigma = 1$ in (6.1) and (6.2), we get the following new results

$$F_{\alpha,\beta}^{(p,\mu,v,\gamma)}(\delta_1,\delta_2,\delta_3;\tau) = \frac{1}{B(\delta_2,\delta_3-\delta_2)}$$
$$\int_{-1}^{1} t^{\delta_2-1}(1-t)^{\delta_3-\delta_2-1}(1-\tau t)^{-\delta_1} E_{\alpha,\beta}^{\gamma}\left(-p\ \frac{2^{\mu+v}}{(1-u)^{\mu}(1-u)^{v}}\right), \qquad (6.4)$$

$$(p \in \mathbb{R}_0^+,\ \alpha,\beta,\gamma,\mu,v \in \mathbb{R}^+;\ and\ \ arg|1-\tau| < \pi,\ Re(\delta_3) > Re(\delta_2) > 0),$$

and

$$\Phi_{\alpha,\beta}^{(p,\mu,v,\gamma)}(\delta_2;\delta_3;\tau) = \frac{1}{B(\delta_2,\delta_3-\delta_2)}$$
$$\times \int_{-1}^{1} t^{\delta_2-1}(1-t)^{\delta_3-\delta_2-1} e^{zt} E_{\alpha,\beta}^{\gamma}\left(-p\ \frac{2^{\mu+v}}{(1-u)^{\mu}(1-u)^{v}}\right), \qquad (6.5)$$

$$(p \in \mathbb{R}_0^+,\ \alpha,\beta,\gamma,\mu,v \in \mathbb{R}^+;\quad Re(\delta_3) > Re(\delta_2) > 0).$$

Theorem 6.2. The following derivative formula for extended Gauss hypergeometric and confluent hypergeometric function holds:

$$\frac{d^n}{d\tau^n}\left\{F_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(\delta_1,\delta_2,\delta_3;\tau)\right\} = \frac{(\delta_1)_n(\delta_2)_n}{(\delta_3)_n}$$
$$\times F_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(\delta_1+n,\delta_2+n;\delta_3+n;\tau), \qquad (6.6)$$

and

$$\frac{d^n}{d\tau^n}\left\{\Phi_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(\delta_2;\delta_3;\tau)\right\} = \frac{(\delta_2)_n}{(\delta_3)_n}\Phi_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(\delta_2+n,\delta_3+n;\tau), \tag{6.7}$$

where $(p \geq 0\,,\ \alpha,\beta,\gamma,\sigma,\mu,v \in \mathbb{R}^+\,;\quad Re(\delta_3) > Re(\delta_2) > 0)$.

Proof. Differentiating (5.1) and (5.2) with respect to τ and using the following formula

$$B(\delta_2,\delta_3-\delta_2) = \frac{\delta_3}{\delta_2}\,B(\delta_2+1,\delta_3-\delta_2)\ \ and\ \ (\delta)_n = \delta(\delta+1)_n. \tag{6.8}$$

we obtain the derivative formulas (6.6) and (6.7) for $n = 1$. Easily applying the same process, we get the desired results (6.6) and (6.7).

Remark 6.2. In case $\alpha = \beta = \sigma = \gamma = \mu = v = 1$ in (6.6) and (6.7), we obtain the corresponding result in [6].

In case $\beta = \sigma = \gamma = \mu = v = 1$ in (6.6) and (6.7), we obtain the corresponding result in [15].
In case $\beta = \sigma = \gamma = \mu = v = 1$ in (6.6) and (6.7), we obtain the corresponding result in [2].
In case $\sigma = \gamma = 1$ and $\mu = v = m$ in (6.6) and (6.7), we obtain the corresponding result in [12].
In case $\sigma = \gamma = 1$ in (6.6) and (6.7), we obtain the corresponding result in [9].

In case $\sigma = 1$ in (6.6) and (6.7), we get the following new results

$$\frac{d^n}{d\tau^n}\left\{F_{\alpha,\beta}^{(p,\mu,v,\gamma)}(\delta_1,\delta_2,\delta_3;\tau)\right\} = \frac{(\delta_1)_n(\delta_2)_n}{(\delta_3)_n} \times F_{\alpha,\beta}^{(p,\mu,v,\gamma)}(\delta_1+n,\delta_2+n;\delta_3+n;\tau), \tag{6.9}$$

and

$$\frac{d^n}{d\tau^n}\left\{\Phi_{\alpha,\beta}^{(p,\mu,v,\gamma)}(\delta_2;\delta_3;\tau)\right\} = \frac{(\delta_2)_n}{(\delta_3)_n}\ \Phi_{\alpha,\beta}^{(p,\mu,v,\gamma)}(\delta_2+n,\delta_3+n;\tau). \tag{6.10}$$

7. Transformation and summation formulas

Theorem 7.1. The following formulas for the extended hypergeometric and con-fluent hypergeometric function hold

$$F_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(\delta_1,\delta_2,\delta_3;\tau) = (1-\tau)^{-\delta_1}\ F_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}\left(\delta_1,\delta_2,\delta_3;\frac{\tau}{1-\tau}\right), \tag{7.1}$$

$$F_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}\left(\delta_1,\delta_2,\delta_3;1-\frac{1}{\tau}\right)=\tau^{\delta_1}\,F_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(\delta_1,\delta_2,\delta_3;1-\tau), \tag{7.2}$$

$$F_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}\left(\delta_1,\delta_2,\delta_3;\frac{\tau}{1+\tau}\right)=(1+\tau)^{\delta_1}\,F_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(\delta_1,\delta_2,\delta_3;-\tau), \tag{7.3}$$

$$\Phi_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(\delta_2;\delta_3;\tau)=e^{\tau}\,\Phi_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(\delta_3-\delta_2;\delta_3;-\tau), \tag{7.4}$$

Proof. Replacing t by $1-t$ and substituting

$$\big(1-\tau(1-t)\big)^{-\delta_1}=(1-\tau)^{-\delta_1}\left(1+\frac{\tau}{1-\tau}\,t\right)^{-\delta_1},$$

in (6.1), we obtain

$$F_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(\delta_1,\delta_2,\delta_3;\tau)=\frac{(1-\tau)^{-\delta_1}}{B(\delta_2,\delta_3-\delta_2)}$$

$$\times\int_{-1}^{1}t^{\delta_2-1}(1-t)^{\delta_3-\delta_2-1}\left(1+\frac{\tau}{1-\tau}\,t\right)^{-\delta_1}E_{\alpha,\beta}^{\gamma,\sigma}\left(-p\,\frac{2^{\mu+v}}{(1-u)^{\mu}(1-u)^{v}}\right), \tag{7.5}$$

$$=\frac{(1-\tau)^{-\delta_1}}{B(\delta_2,\delta_3-\delta_2)}$$

$$\times\int_{-1}^{1}t^{\delta_2-1}(1-t)^{\delta_3-\delta_2-1}\left(1-\frac{-\tau}{1-\tau}\,t\right)^{-\delta_1}E_{\alpha,\beta}^{\gamma,\sigma}\left(-p\,\frac{2^{\mu+v}}{(1-u)^{\mu}(1-u)^{v}}\right). \tag{7.6}$$

In view of (6.1), we get the desired result (7.1).

Replacing τ by $1-\frac{1}{\tau}$ and $\frac{\tau}{1+\tau}$ in (7.1) yield (7.2) and (7.3) respectively. Similarly as (7.1), we can establish (7.4).

Remark 7.1. In case $\alpha=\beta=\sigma=\gamma=\mu=v=1$ in (7.1) and (7.4), we obtain the corresponding result in [6].

In case $\sigma=\mu=v=1$ in (7.1) to (7.4), we obtain the corresponding result in [2].

In case $\beta=\sigma=\gamma=\mu=v=1$ in (7.1) to (7.4), we obtain the corresponding result in [15].

In case $\beta=\sigma=\gamma=\mu=v=1$ in (7.1) to (7.4), we obtain the corresponding result in [6].

In case $\sigma=\gamma=1$ and $\mu=v=m$ in (7.1) to (7.4), we obtain the corresponding result in [12].

In case $\sigma=\gamma=1$ in (7.1) to (7.4), we obtain the corresponding result in [9].

In case $\sigma=1$ in (7.1) to (7.4), we get the following new results

$$F_{\alpha,\beta}^{(p,\mu,v,\gamma)}(\delta_1,\delta_2,\delta_3;\tau) = (1-\tau)^{-\delta_1} F_{\alpha,\beta}^{(p,\mu,v,\gamma)}\left(\delta_1,\delta_2,\delta_3;\frac{\tau}{1-\tau}\right), \tag{7.7}$$

$$F_{\alpha,\beta}^{(p,\mu,v,\gamma)}\left(\delta_1,\delta_2,\delta_3;1-\frac{1}{\tau}\right) = \tau^{\delta_1} F_{\alpha,\beta}^{(p,\mu,v,\gamma)}(\delta_1,\delta_2,\delta_3;1-\tau), \tag{7.8}$$

$$F_{\alpha,\beta}^{(p,\mu,v,\gamma)}\left(\delta_1,\delta_2,\delta_3;\frac{\tau}{1+\tau}\right) = (1+\tau)^{\delta_1} F_{\alpha,\beta}^{(p,\mu,v,\gamma)}(\delta_1,\delta_2,\delta_3;-\tau), \tag{7.9}$$

and

$$\Phi_{\alpha,\beta}^{(p,\mu,v,\gamma)}(\delta_2;\delta_3;\tau) = e^{\tau}\, \Phi_{\alpha,\beta}^{(p,\mu,v,\gamma)}(\delta_3-\delta_2;\delta_3;-\tau), \tag{7.10}$$

Theorem 7.2. The following summation formula hold

$$F_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(\delta_1,\delta_2,\delta_3;1) = \frac{B_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(\delta_2,\delta_3-\delta_1-\delta_2)}{B(\delta_2,\delta_3-\delta_2)}, \tag{7.11}$$

$$(p \in \mathbb{R}_0^+,\ \alpha,\beta,\gamma,\sigma,\mu,v \in \mathbb{R}^+;\quad Re(\delta_3-\delta_1-\delta_2)>0).$$

Proof. Putting $\tau = 1$ in (6.1) and using the definition (2.1), we obtain the desir-ed result (7.11).

Remark 7.2. In case $\alpha=\beta=\sigma=\gamma=\mu=v=1$, with $p=0$ in (7.11), we obtain the Gauss summation formula ${}_2F_1$

$${}_2F_1(\delta_1,\delta_2,\delta_3;1) = \frac{\Gamma(\delta_3)\,\Gamma(\delta_3-\delta_1-\delta_2)}{\Gamma(\delta_3-\delta_1)\Gamma(\delta_3-\delta_2)},\ (Re(\delta_3-\delta_1-\delta_2)>0)\,. \tag{7.12}$$

8. A generating function for $F_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(\delta_1,\delta_2,\delta_3;\tau)$

Theorem 8.1. The following generating function for $F_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(\delta_1,\delta_2,\delta_3;\tau)$ hold

$$\sum_{n=k}^{\infty}(\delta_1)_n F_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(\delta_1+k,\delta_2,\delta_3;\tau)\ \frac{\tau^k}{k!}$$
$$= (1-t)^{-\delta_1} F_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}\left(\delta_1+k,\delta_2,\delta_3;\frac{\tau}{1-t}\right), \tag{8.1}$$

$$(p \in \mathbb{R}_0^+,\ |t|<1,\ \alpha,\beta,\gamma,\sigma,\mu,v \in \mathbb{R}^+).$$

Proof. Let Δ be the left hand side (L.H.S) of (8.1). From (5.1), we have

$$\Delta = \sum_{k=0}^{\infty}(\delta_1)_k \left(\sum_{n=0}^{\infty}\frac{(\delta_1+k)_n\, B_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(\delta_2+n,\delta_3-\delta_2)}{B(\delta_2,\delta_3-\delta_2)}\,\frac{\tau^n}{n!}\right)\frac{t^k}{k!} \tag{8.2}$$

$$= \sum_{k=0}^{\infty} (\delta_1)_k \ \frac{B_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(\delta_2 + n, \delta_3 - \delta_2)}{B(\delta_2, \delta_3 - \delta_2)} \left(\sum_{n=0}^{\infty} (\delta_1 + k)_n \ \frac{t^k}{k!} \right) \frac{\tau^n}{n!}$$

$$= (1 - t)^{-\delta_1} \sum_{k=0}^{\infty} (\delta_1)_k \ \frac{B_{\alpha,\beta}^{(p,\mu,v,\gamma,\sigma)}(\delta_2 + n, \delta_3 - \delta_2)}{B(\delta_2, \delta_3 - \delta_2)} \left(\frac{\tau}{1 - t} \right)^n \frac{1}{n!}. \qquad (8.3)$$

Finally by using (5.1) in (8.3), we get the right side of (8.1).

Remark 8.1. In case $\alpha = \beta = \sigma = \gamma = \mu = v = 1$ in (8.1), we obtain the corresponding result in [6].

In case $\beta = \sigma = \gamma = \mu = v = 1$ in (8.1), we obtain the corresponding result in [15].
In case $\sigma = \mu = v = 1$ in (8.1), we obtain the corresponding result in [2].
In case $\sigma = \gamma = 1$ in (8.1), we obtain the corresponding result in [9].

In case $\sigma = \gamma = 1$ and $\mu = v = m$ in (8.1), we obtain the corresponding result in [12].
In case $\sigma = 1$ in (8.1), we get the following new result

$$\sum_{n=k}^{\infty} (\delta_1)_n \, F_{\alpha,\beta}^{(p,\mu,v,\gamma)}(\delta_1 + k, \delta_2, \delta_3; \tau) \ \frac{\tau^k}{k!}$$
$$= (1 - t)^{-\delta_1} F_{\alpha,\beta}^{(p,\mu,v,\gamma)} \left(\delta_1 + k, \delta_2, \delta_3; \frac{\tau}{1-t} \right). \qquad (8.4)$$

References

[1] Al-Gonah A. A. and Mohammed W. K., A new extension of extended Gamma and Beta functions and their properties, Journal of Scientific and Engineering Research 5(9) (2018), 257-270.

[2] Al-Gonah A. A. and Mohammed W. K., A new forms of extended of extended and their properties, Journal of Engineering and Applied Science letters Research 4(1) (2021), 30-41.

[3] Andrews G. E., Askey R., Roy R., Special functions, Cambridge University Press, Cambridge, 1999.

[4] Chaudhry M. A., Zubair S. M., Generalized incomplete gamma functions with applications, J. Comput. Appl. Math., 1994. 55: p. 99–124.

[5] Chaudhry M. A., Qadir A., Rafique M. and Zubair S. M., Extension of Euler's Beta function, J. Comput. Appl. Math., 1997. 78: p. 19–32.

[6] Chaudhry M. A., Qadir A., Srivastava H. M. and Paris R. B., Extended Hypergeometric and Confluent Hypergeometric functions, Appl. Math. Comput., 2004. 159: p. 589–602.

[7] Choi J., Rathie A. K., Parmar R. K., Extension of extended Beta , hypergeometric and confluent hypergeometric functions, Honam Mathematical J., 2014. 36(2): p. 357–385.

[8] Ghayasuddin M., Khan N. U., Remarks on extended Gauss hypergeometric functions, Acta Universitatis Apulensis, 2017. 49: 1–13.

[9] Khan N. and Husain S., A note on extended Beta function involving generalized Mittag-Leffler function and its applications, Twms J. App. and Eng. Math. 2022. 12(1): p. 71-81.

[10] Khan N. U., Usman T., Aman M., Extended Beta, Hypergeometric and confluent Hypergeometric functions, Transactions issues Mathematics series of physical-technical Mathematics science Azerbaijan National Academy of Science, 2019. 39(1), 83–97.

[11] Mittag-Leffler G. M., Sur la nouvelle function Eα(z), C. R. Acad. Sci. Paris, 1903. 137: p. 554-558.

[12] Oraby K., Rizq A., Ahmed M., Khaled M., Ahmed E. and M. Magdy, "Generalization of Beta functions in terms of Mittag-Leffler function", Frontiers in Scientific Research and Technology, (2020). vol.1, pp.81-88.

[13] Özarslan M. A. and Özerjin E., Some generating relation for extended hypergeometric functions via generalized fractional derivative operator, Mathematical and Computer Modelling, 2010. 52: p. 1825-1833.

[14] Özerjin E., Özarslan M. A., Altin A., Extension of gamma, Beta and hypergeometric functions, J. Comput. Appl. Math., 2011. 235: p. 4601–4610.
[15] Shadab M., Jabee S., Choi J., An extended Beta function and its applications, Far East Journal of Mathematical Sciences, 2018. 103(1): p. 235–251.
[16] Shukla A.K. and Prajapati J.C., On a generalization of Mittag-Leffler function and its properties, J. Math. Anal. Appl., 2007. 336: p. 797-811.
[17] Srivastava H. M. and Manocha H. L., A Treatise on Generating Functions, New York: Halsted Press, 1984.

STARDOM UNIVERSITY

STARDOM UNIVERSITY

Stardom Scientific Journal of

Natural and Engineering Sciences

STARDOM SCIENTIFIC JOURNAL OF NATURAL AND ENGINEERING SCIENCES
PUBLISHED TWICE A YEAR BY STARDOM UNIVERSITY
Volume 2 - 1st issue 2024
International deposit number : ISSN 2980-3756